Pattern Statistics on Restricted Random Permutations

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US Naval Academy
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Agenda

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Introduction to Permutation Patterns

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Introduction to Analytic Combinatorics

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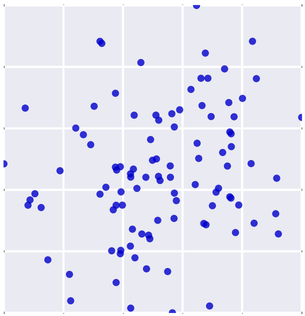
Pattern Statistics in Restricted Permutations

Random Data



Permutations

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Definition

An *permutation of length n* is a bijection from the set $[n] = \{1, 2, \ldots n\}$ to itself. The *one-line notation* for a permutation π is

$$\pi = \pi(1)\pi(2)\ldots\pi(n).$$

The set of all permutations of length n is denoted $\mathfrak{S}_n$.

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Examples

- The sequence $\pi = 5172643$ is a permutation of length 7.

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Examples

- The sequence $\pi = 5172643$ is a permutation of length 7.
- The six permutations of length 3 are

$$\mathfrak{S}_3 = \{123, 132, 213, 231, 312, 321\}.$$

Plotting Permutations

Definition

If π is a permutation of length n, then the *plot* of π is the set of points

$$\{(1, \pi(1)), (2, \pi(2)), \cdots (n, \pi(n))\} \subset \mathbb{R}^2$$

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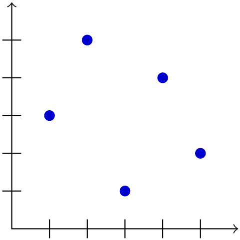

$\pi = 35142$

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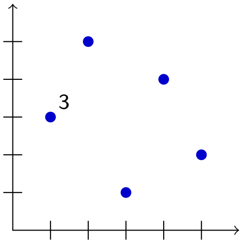

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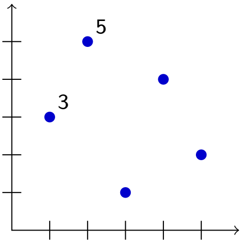

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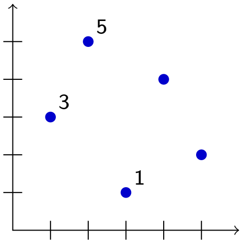

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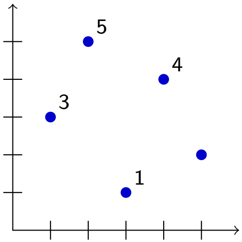

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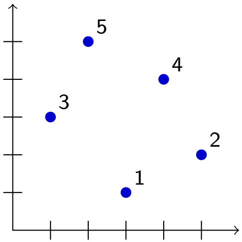

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Dots on a Plane

Definition

Let A and B be two sets of n points in $\mathbb{R}^2$, each with the property that no two points lie on the same horizontal or vertical line.
Say that A is *order isomorphic* to B (denoted $A \sim B$) if A can be transformed into B by stretching, contracting, and translating the axes horizontally and vertically.

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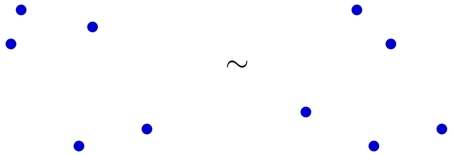

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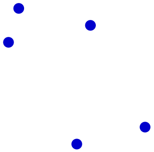

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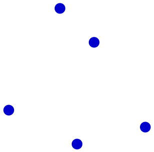

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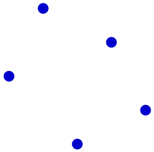

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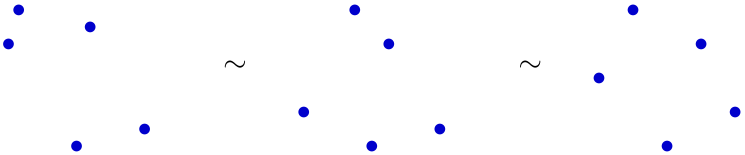

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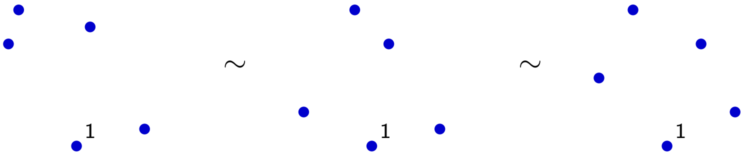

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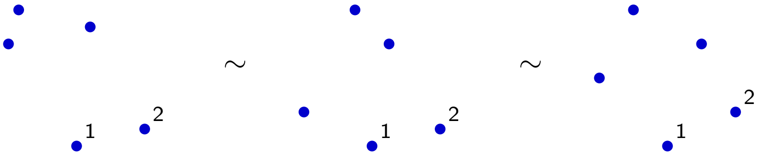

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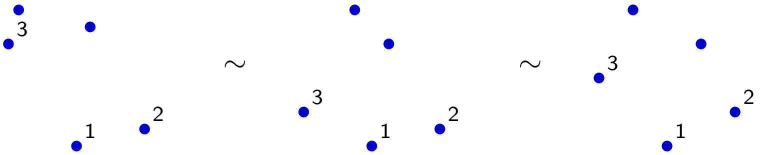

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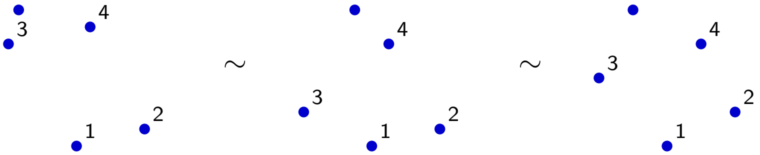

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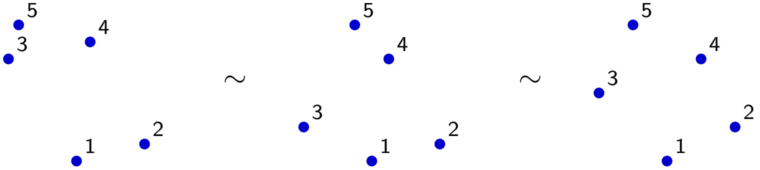

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Permutation Patterns

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Definition

Let $\pi = \pi(1)\pi(2)\cdots\pi(n)$ and $\sigma = \sigma(1)\sigma(2)\cdots\sigma(k)$ be two permutations. π *contains* σ *as a pattern* (written $\sigma \prec \pi$) if there is some subsequence $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ which is order isomorphic to the entries of σ (i.e., $\pi(i_j) < \pi(i_k)$ if and only if $\sigma(j) < \sigma(k)$).

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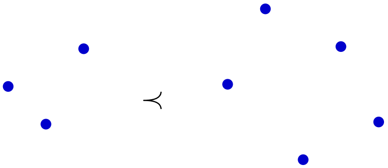

$213 \qquad \prec \qquad 35142$

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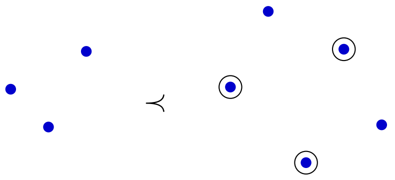

$213 \quad \prec \quad \mathbf{3}5\mathbf{1}\mathbf{4}2$

Example

The pattern 12 is contained in all permutations *except* for the decreasing ones:

$$12 \not\preceq n \ldots 321.$$

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$$12 \not\prec n \dots 321.$$

Definition

If a permutation π does not contain a pattern σ, we say that π *avoids* σ. The set of all permutations which avoid a given pattern (or set of patterns) σ is denoted

$$\mathrm{Av}(\sigma).$$

The Pattern Poset

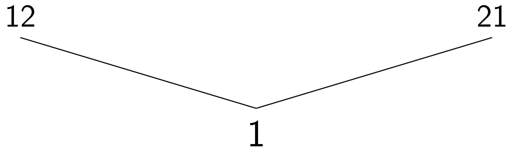
1

12

21

1

The Pattern Poset



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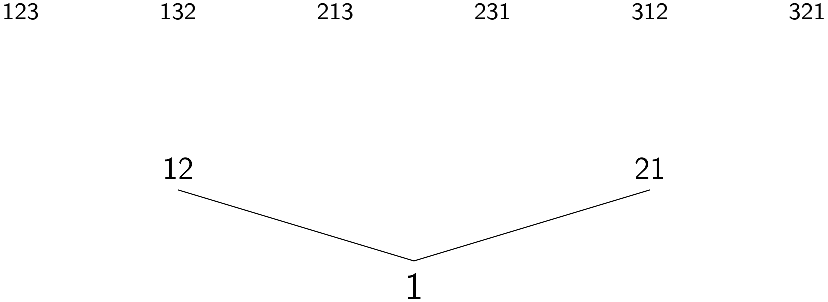

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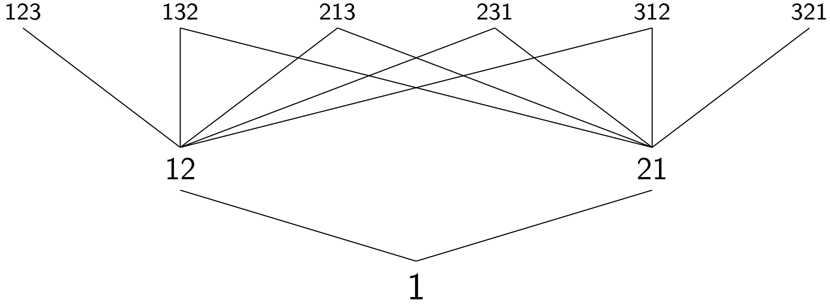

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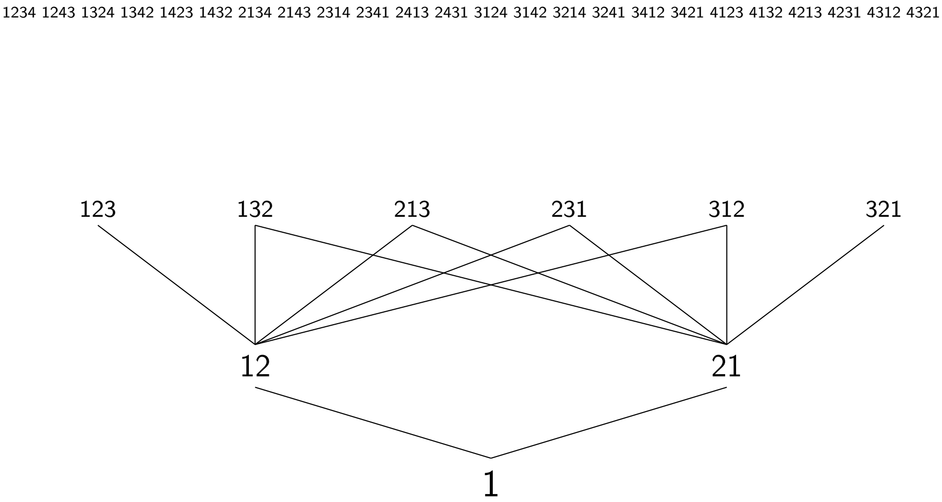

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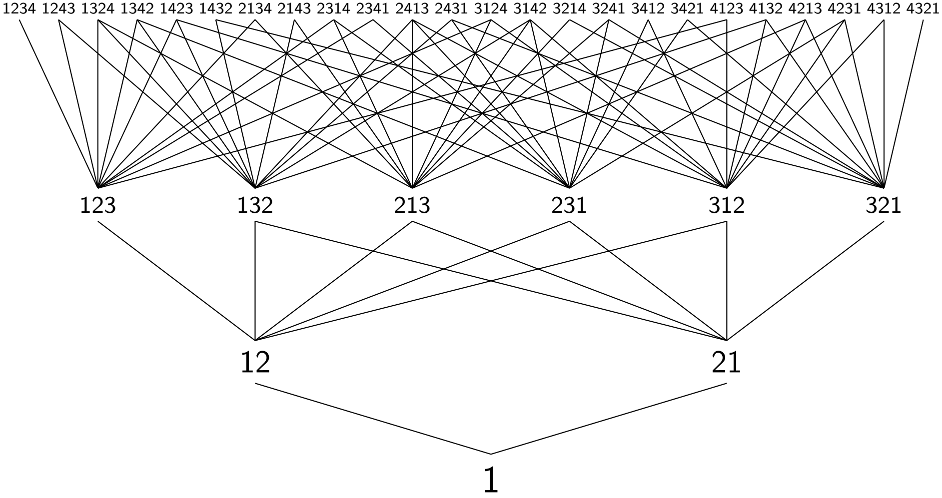

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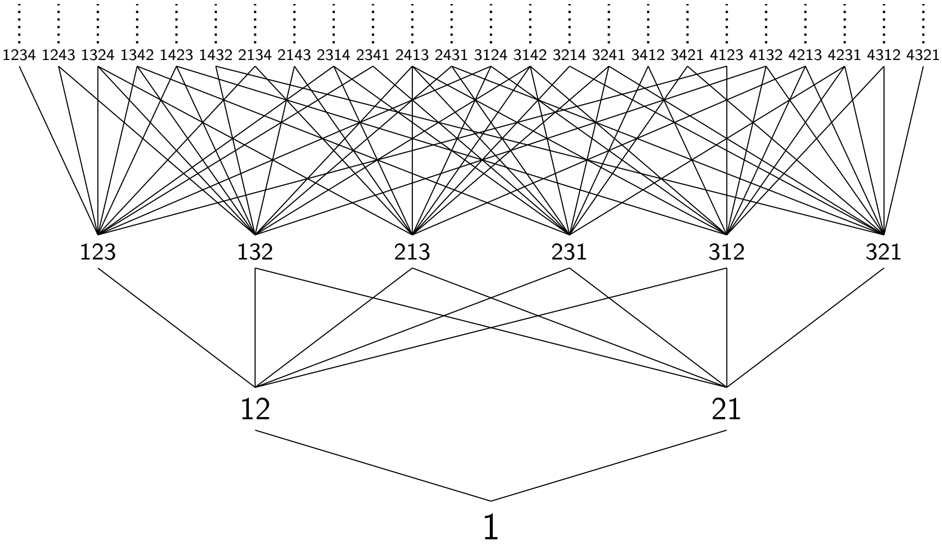

Pattern Avoidance

Question

Given a pattern, how many permutations (of length n) avoid that pattern?

Enumerations of specific permutation classes

From Wikipedia, the free encyclopedia

In the study of permutation patterns, there has been considerable interest in enumerating specific permutation classes, especially those with relatively few basis elements.

Contents [show]

Classes avoiding one pattern of length 3 [edit]

There are two symmetry classes and a single Wilf class for single permutations of length three.

β	sequence enumerating $Av_n(\beta)$	OEIS	type of sequence	exact enumeration reference
123 231	1, 2, 5, 14, 42, 132, 429, 1430, ...	A000108	algebraic (nonrational) g.f. Catalan numbers	MacMahon (1916) Knuth (1968)

Classes avoiding one pattern of length 4 [edit]

There are seven symmetry classes and three Wilf classes for single permutations of length four.

β	sequence enumerating $Av_n(\beta)$	OEIS	type of sequence	exact enumeration reference
1342 2413	1, 2, 6, 23, 103, 512, 2740, 15485, ...	A022558	algebraic (nonrational) g.f.	Bóna (1997)
1234 1243 1432 2143	1, 2, 6, 23, 103, 513, 2761, 15767, ...	A005802	holonomic (nonalgebraic) g.f.	Gessel (1990)
1324	1, 2, 6, 23, 103, 513, 2762, 15793, ...	A061552		

Classes avoiding two patterns of length 4 [edit]

There are 56 symmetry classes and 38 Wilf equivalence classes. Only 8 of these remain unenumerated, and the generating functions for 3 of those 8 classes are conjectured not to satisfy any algebraic differential equation (ADE) by Albert et al. (preprint); in particular, their conjecture would imply that the generating functions are not D-finite.

B	sequence enumerating $Av_n(B)$	OEIS	type of sequence	exact enumeration reference
4321, 1234	1, 2, 6, 22, 86, 306, 882, 1764, ...	A206736	finite	Erdős–Szekeres theorem
4312, 1234	1, 2, 6, 22, 86, 321, 1085, 3266, ...	A116705	polynomial	Kremer & Shiu (2003)
4321, 3124	1, 2, 6, 22, 86, 330, 1198, 4087, ...	A116708	rational g.f.	Kremer & Shiu (2003)
4312, 2134	1, 2, 6, 22, 86, 330, 1206, 4174, ...	A116706	rational g.f.	Kremer & Shiu (2003)
4321, 1324	1, 2, 6, 22, 86, 332, 1217, 4140, ...	A165524	polynomial	Vatter (2012)
4321, 2143	1, 2, 6, 22, 86, 333, 1235, 4339, ...	A165525	rational g.f.	Albert, Atkinson & Brignall (2012)
4312, 1324	1, 2, 6, 22, 86, 335, 1266, 4598, ...	A165526	rational g.f.	Albert, Atkinson & Brignall (2012)
4231, 2143	1, 2, 6, 22, 86, 335, 1271, 4680, ...	A165527	rational g.f.	Albert, Atkinson & Brignall (2011)
4231, 1324	1, 2, 6, 22, 86, 336, 1282, 4758, ...	A165528	rational g.f.	Albert, Atkinson & Vatter (2009)
4213, 2341	1, 2, 6, 22, 86, 336, 1290, 4870, ...	A116709	rational g.f.	Kremer & Shiu (2003)

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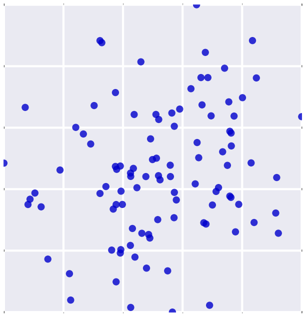

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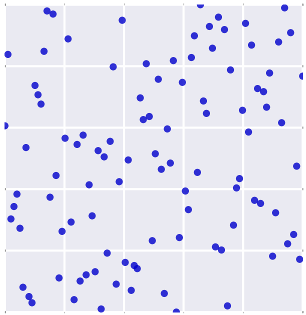

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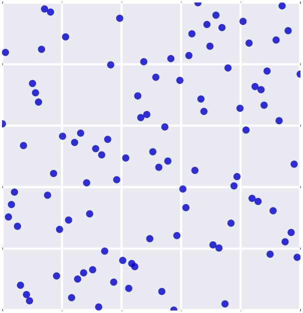

$\pi =$ 61 84 31 35 39 28 9 54 6 4 74 71 68 85 98 38 97 45 12 27 57 89 30 5 55 11 58
13 42 32 14 53 2 51 20 56 80 10 43 95 17 50 8 16 15 70 63 81 64 24 52 76 47
7 60 49 82 1 25 75 40 34 83 90 46 100 69 65 93 86 22 96 21 92 3 79 29 41
44 66 94 59 87 37 73 36 72 67 78 19 33 88 62 99 23 91 26 48 18 77

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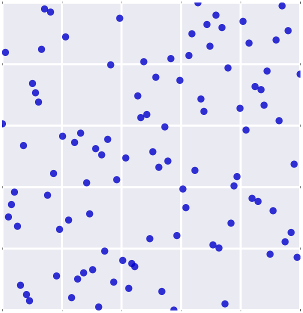

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44 66 94 59 87 37 73 36 72 67 78 19 33 88 62 99 23 91 26 48 18 77

So What?

Combinatorial Classes

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Idea

Every combinatorial object is just some underlying (typically finite) set with some structure imposed on it.

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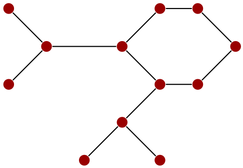
Every combinatorial object is just some underlying (typically finite) set with some structure imposed on it.

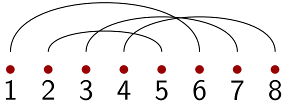
Definition

A combinatorial class is a set of objects together with a (non-negative-integer valued) *size* function, with the property that there are finitely many objects of each size

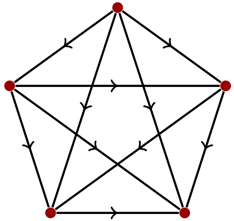
Combinatorial Classes



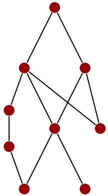
Graphs



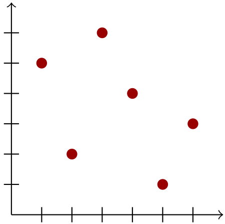
Matchings



Tournaments



Posets



Permutations

Operations on Classes

Idea

Classes can be combined by various operations, including union (denoted $\cup$) and cartesian product (denoted $\cdot$).

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Classes can be combined by various operations, including union (denoted $\cup$) and cartesian product (denoted $\cdot$).

Examples

Let $\mathcal{G}$ denote the class of all (unlabelled, undirected, simple) graphs, and let $\mathcal{C}$ denote the (sub)class of non-empty connected graphs. Then

$$\mathcal{G} = \mathcal{C} \cup (\mathcal{C} \cdot \mathcal{C}) \cup (\mathcal{C} \cdot \mathcal{C} \cdot \mathcal{C}) \cup \ldots$$

Generating Functions

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(Loose) Definition

For a class $\mathcal{A}$, the *generating function* for $\mathcal{A}$ is the function $A = \sum_{n \geq 0} a_n z^n$, where a_n is the number of distinct objects within the class on an underlying set of size n.

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Example

For the class $\mathcal{S}$ of permutations, the generating function is

$$\sum_{n \geq 0} n! z^n.$$

Theorem

If $\mathcal{A}$ and $\mathcal{B}$ are combinatorial classes with generating funtions $F_A(z)$ and $F_B(z)$. Then

$$F_{A\cup B}(z) = A(z) + B(z) = \sum_{n\geq 0}(a_n + b_n)z^n$$

and

$$F_{A\cdot B}(z) = A(z)\cdot B(z) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \ldots$$

Theorem

Letting $\mathcal{G}$ be the class of all graphs, and $\mathcal{C}$ be the class of connected non-empty graphs, we have

$$\mathcal{G} = \mathcal{C} \cup (\mathcal{C} \cdot \mathcal{C}) \cup (\mathcal{C} \cdot \mathcal{C} \cdot \mathcal{C}) + \dots$$
$$G = C + C^2 + C^3 + \dots$$
$$= \frac{C}{1 - C},$$

and also

$$C = \frac{G}{G + 1}.$$

1. The main constructions of disjoint union (combinatorial sum), product, sequence, powerset, multiset, and cycle and their translation into generating functions (Theorem I.1).

Construction		*OGF*
Union	$\mathcal{A} = \mathcal{B} + \mathcal{C}$	$A(z) = B(z) + C(z)$
Product	$\mathcal{A} = \mathcal{B} \times \mathcal{C}$	$A(z) = B(z) \cdot C(z)$
Sequence	$\mathcal{A} = \mathrm{SEQ}(\mathcal{B})$	$A(z) = \dfrac{1}{1 - B(z)}$
Powerset	$\mathcal{A} = \mathrm{PSET}(\mathcal{B})$	$A(z) = \exp\left(B(z) - \frac{1}{2}B(z^2) + \cdots\right)$
Multiset	$\mathcal{A} = \mathrm{MSET}(\mathcal{B})$	$A(z) = \exp\left(B(z) + \frac{1}{2}B(z^2) + \cdots\right)$
Cycle	$\mathcal{A} = \mathrm{CYC}(\mathcal{B})$	$A(z) = \log \dfrac{1}{1 - B(z)} + \dfrac{1}{2} \log \dfrac{1}{1 - B(z^2)} + \cdots$

– *Analytic Combinatorics*, Phillipe Flajolet and Robert Sedgewick

Dyck Paths

Definition

A *Dyck path of length n* is a path from $(0, 0)$ to $(2n, 0)$ using the steps $(1, 1)$ and $(1, -1)$, which never goes below the x-axis.

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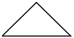

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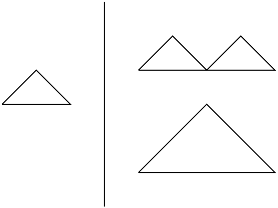

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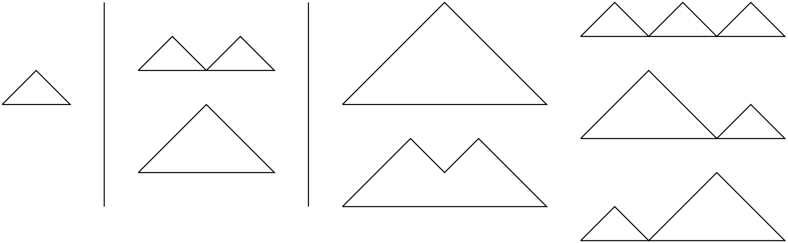

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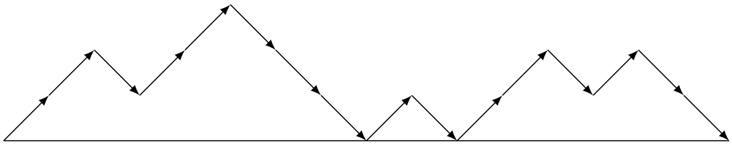

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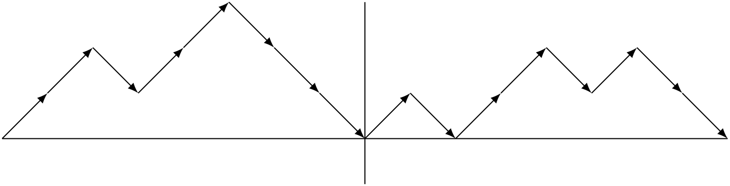

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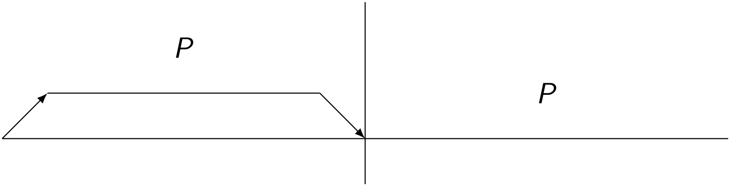

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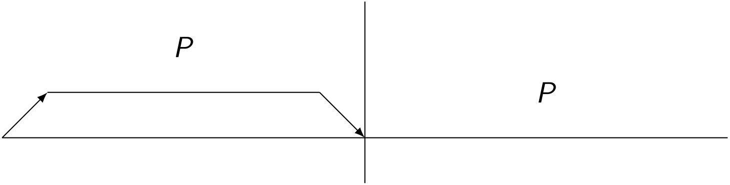

$$P = zP^2 + 1$$

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The radius of convergence $(1/4)$ gives that, roughly,

$$p_n \approx 4^n.$$

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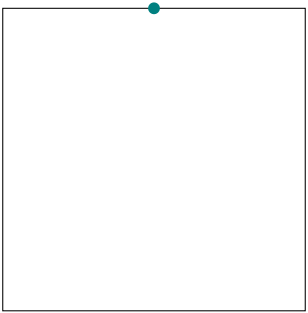

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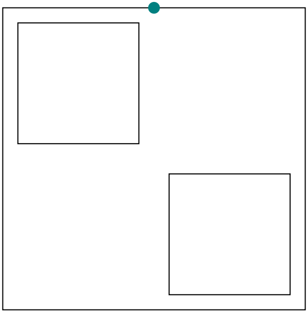

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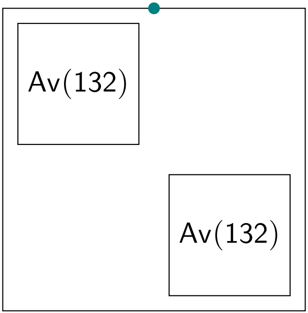

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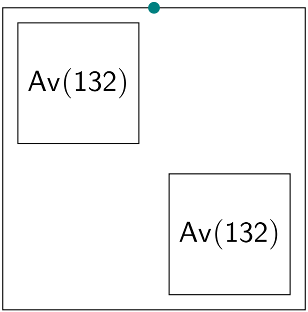

Theorem

The 132-avoiding permutations are in bijection with Dyck paths. (These numbers are called the *Catalan numbers*.)

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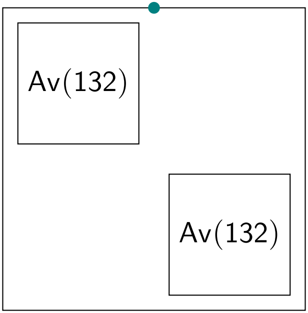

Theorem

The 132-avoiding permutations are in bijection with Dyck paths. (These numbers are called the *Catalan numbers*.)

Corollary

Almost all permutations contain 132.

Pattern Occurrences

Counting Patterns

The number of occurrences of σ in π (denoted $\nu_\sigma(\pi)$) is the number of σ-isomorphic subsets.

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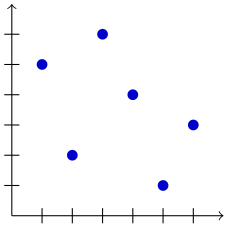

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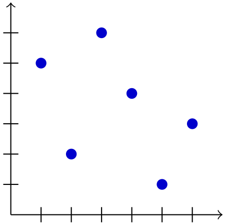

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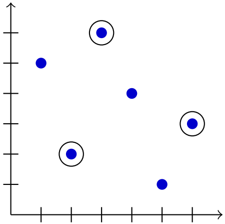

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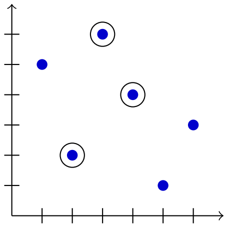

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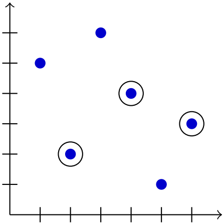

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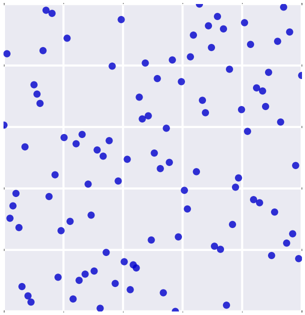

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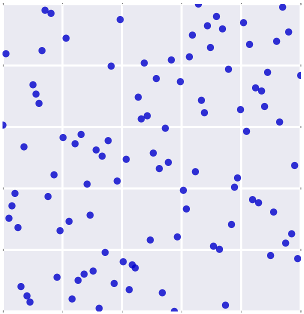

ν_{12}	ν_{21}	Avg
2803	2147	2475

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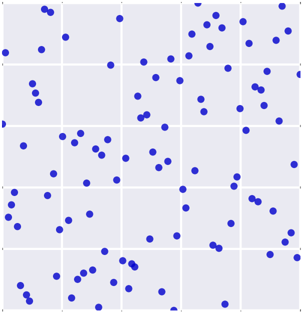

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ν_{123}	ν_{132}	ν_{213}	ν_{231}	ν_{312}	ν_{321}	Avg
35357	30063	31414	22321	23348	19197	26950

Patterns as Random Variables

Theorem (Bóna 2007)

For a (uniformly) randomly selected permutation of length n, the random variables ν_σ are asymptotically normal as n approaches infinity.

Theorem (Janson, Nakamura, Zeilberger 2013)

For a randomly selected permutation of length n and two patterns σ and ρ, the random variables ν_σ and ν_ρ are asymptotically jointly normally distributed as $n \to \infty$.

Linearity of Expectation

Theorem

Let $|\sigma| = k$ In a randomly chosen n-permutation,

$$\mathbb{E}\left[\nu_\sigma\right] = \binom{n}{k}\frac{1}{k!}.$$

Motivation

Corollary

In $\mathfrak{S}_n$, the number of occurrences of a specific pattern depends only on the length of the pattern. That is, for a pattern $\sigma \in \mathfrak{S}_k$, we have

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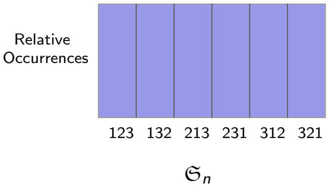

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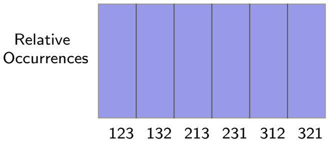
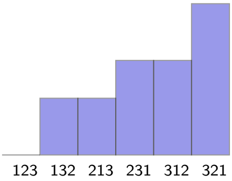


$\mathfrak{S}_n$



$\mathrm{Av}_n\,123$

Main Theorems

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Theorem (Bóna 2010)

In $\mathrm{Av}_n\,132$ we have

$$\nu_{123} \sim \frac{n^2}{2}$$
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Theorem (Albert, H, Pantone 2014)

The equipopularity classes of the separable permutations (a superclass of $\mathrm{Av}\,132$) are in bijection with integer partitions (and also we can count them based on the partition).

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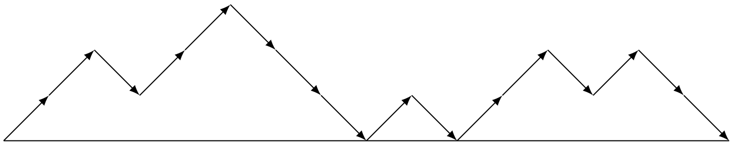

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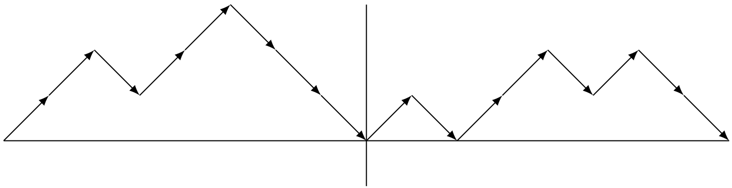

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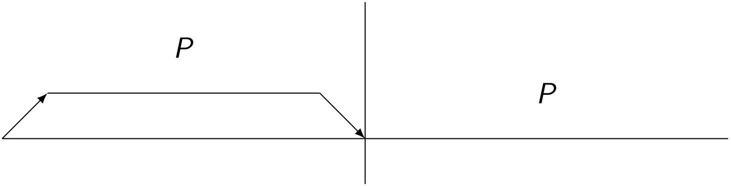

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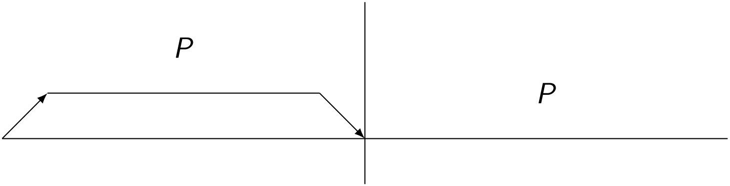

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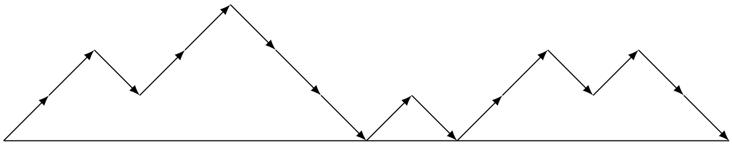

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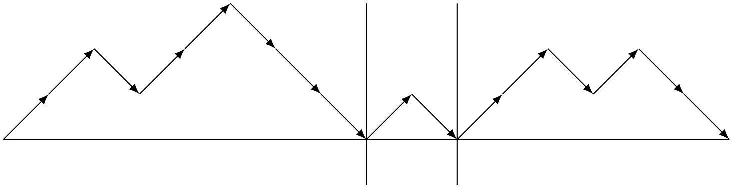

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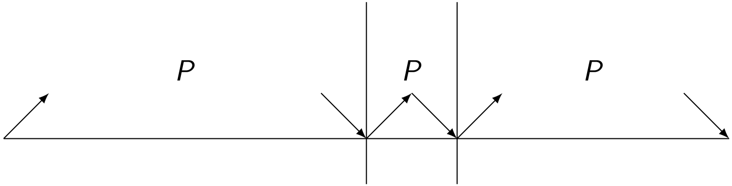

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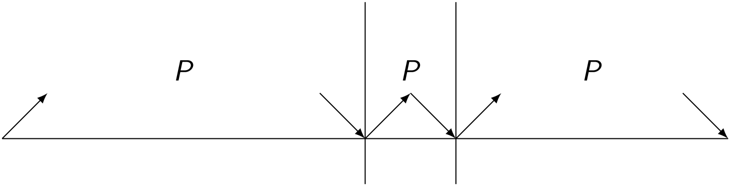

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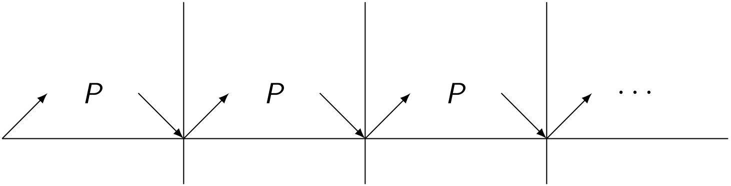

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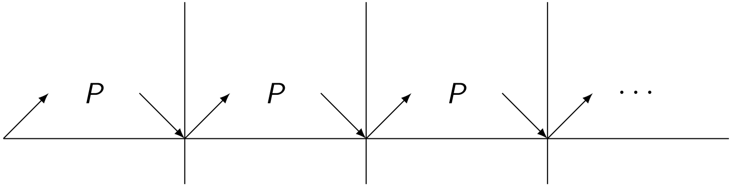

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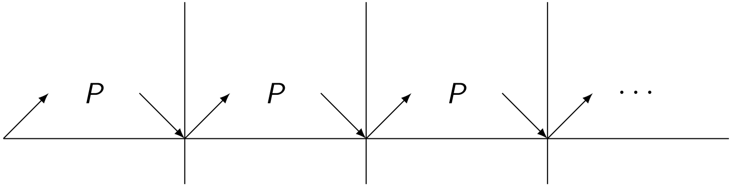

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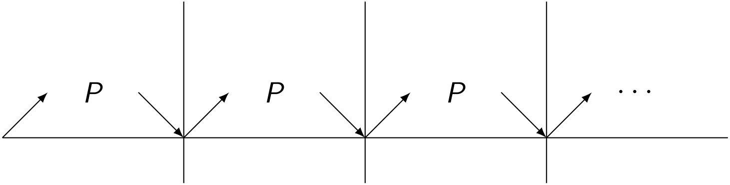

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Multivariate Generating Functions

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Idea

We've been using z to record the size of an object. We can also use other variables to mark other statistics.

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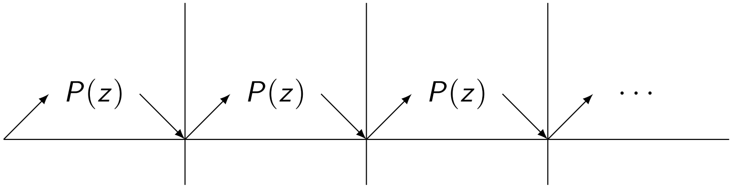
Case Study

Given a random Dyck path, what is the expected number times it touches the axis?

Dyck Path Returns



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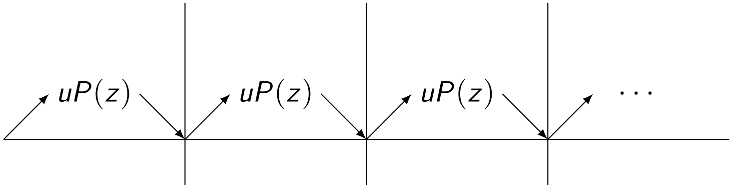

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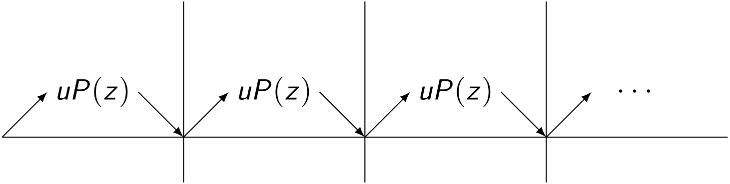

$$
\begin{aligned}
P(z,u) &= zuP(z,1) + z^2u^2P(z,1)^2 + u^3z^3P(z,1)^3 + \ldots \\
&= \frac{zuP(z,1)}{1 - zuP(z,1)} \\
&= u\frac{1-\sqrt{1-4z}}{2-u+u\sqrt{1-4z}}
\end{aligned}
$$

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```
In [1]: from sympy import var, sqrt
In [2]: z, u, f, fz1 = var('z u f fz1')
In [3]:
In [3]: fz1 = (1 - sqrt(1-4*z))/(2*z)
In [4]: fz1.series(z, 0, 6)
Out[4]: 1 + z + 2*z**2 + 5*z**3 + 14*z**4 + 42*z**5 + O(z**6)
In [5]:
In [5]: f = (z*u*fz1)/(1 - z*u*fz1)
In [6]: f.series(z,0,4)
Out[6]: z**2*(u**2 + u) + z**3*(u**3 + 2*u**2 + 2*u) + u*z + O(z**4)
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In [3]: fz1 = (1 - sqrt(1-4*z))/(2*z)
In [4]: fz1.series(z, 0, 6)
Out[4]: 1 + z + 2*z**2 + 5*z**3 + 14*z**4 + 42*z**5 + O(z**6)
In [5]:
In [5]: f = (z*u*fz1)/(1 - z*u*fz1)
In [6]: f.series(z,0,4)
Out[6]: z**2*(u**2 + u) + z**3*(u**3 + 2*u**2 + 2*u) + u*z + O(z**4)
```

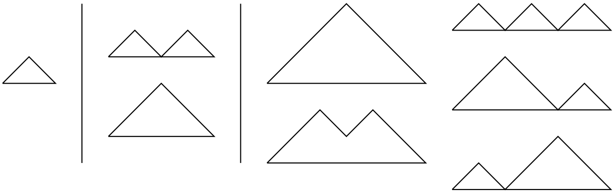

Expectation

Theorem

If r_n is the total number of returns across all length $2n$ Dyck paths, then

$$\begin{aligned}\sum_{n\geq 0} r_n z^n &= \partial_u P(z,u)\Big|_{u=1} \\ &= \frac{1-\sqrt{1-4z}}{1-2z+\sqrt{1-4z}} \\ &= z + 3z^2 + 9z^3 + 28z^4 + 90z^5 + \ldots \quad = \frac{3(2n)!}{(n+2)!(n-1)!}\end{aligned}$$

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$$= z + 3z^2 + 9z^3 + 28z^4 + 90z^5 + \ldots \quad = \frac{3(2n)!}{(n+2)!(n-1)!}$$

Corollary

The expected number of runs in a randomly selected Dyck path of length n is

$$\frac{3n}{n+2}.$$

Higher Moments

Idea

The nth factorial moment can be calculated by taking successive derivatives of the bivariate generating function.

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Theorem

The variance of the number of runs in a random Dyck path of semilength n is

$$\frac{2(2n+1)}{(n+2)} - \frac{5(2n+3)(2n+1)}{2(n+3)(n+2)} + \frac{16(2n+5)(2n+3)(2n+1)}{(n+4)(n+3)(n+2)} - \frac{9n^2}{(n+2)^2}$$

Other Objects

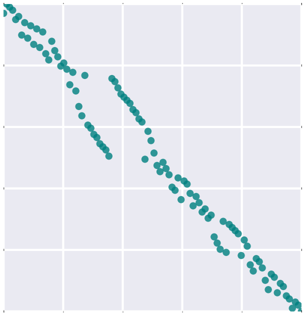
This translates immediately to any other object with the same recursive structure:

$$\mathcal{P} = \mathcal{Z} \cdot \mathcal{P} \cup (\mathcal{Z} \cdot \mathcal{P})^2 \cup (\mathcal{Z} \cdot \mathcal{P})^3 \cup (\mathcal{Z} \cdot \mathcal{P})^4 \ldots$$

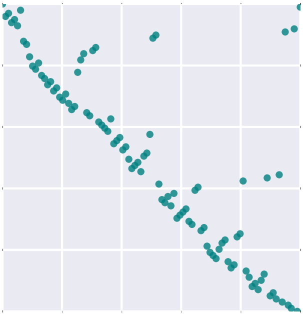
Random Restricted Data

Random Restricted Data

Random 123-avoider



Random 132-avoider



Data

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
4	10	0	11	11	11	13
5	68	0	81	81	81	109
6	392	0	500	500	500	748
7	2063	0	2794	2794	2794	4570

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
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Av 123

length	123	132	213	231	312	321
3	0	1	1	1	1	1
4	0	9	9	11	11	16

Data

Av 132

length	123	132	213	231	312	321
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Av 123

length	123	132	213	231	312	321
3	0	1	1	1	1	1
4	0	9	9	11	11	16
5	0	57	57	81	81	144
6	0	312	312	500	500	1016
7	0	1578	1578	2794	2794	6271

Counting Patterns within Av 132

Sketch of proof:

Counting Patterns within Av 132

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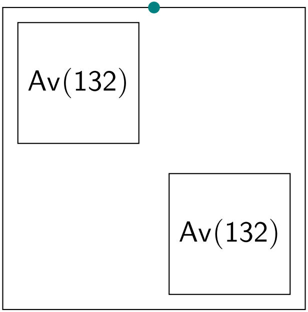
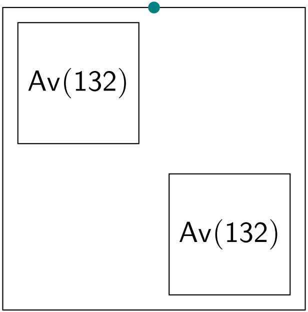


Counting Patterns within Av 132

Sketch of proof:



□

Patterns Within Av(123)

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Theorem (H 2012)

The total nuber of 231 (and 312) patterns is identical within the sets $\mathrm{Av}_n(123)$ and $\mathrm{Av}_n(132)$.

Patterns Within Av(123)

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The total nuber of 231 (and 312) patterns is identical within the sets $\mathrm{Av}_n(123)$ and $\mathrm{Av}_n(132)$.
Further, within $\mathrm{Av}_n(123)$,

$$\nu_{132} = \nu_{213} \sim \sqrt{\frac{n}{\pi}} 4^n,$$

$$\nu_{231} = \nu_{312} \sim \frac{n}{2} 4^n,$$

$$\text{and} \quad \nu_{321} \sim \frac{8}{3} \sqrt{\frac{n^3}{\pi}} 4^n.$$

Sketch of Proof: Patterns in Av(123)

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ν_{132} ν_{213} ν_{231} ν_{312} ν_{321}

Sketch of Proof: Patterns in Av(123)

$$\nu_{132} + \nu_{213} + \nu_{231} + \nu_{312} + \nu_{321} = \binom{n}{3} c_n$$

(Both sides count the number of length three patterns)

Sketch of Proof: Patterns in Av(123)

$$2\nu_{132} + 2\nu_{213} + \nu_{231} + \nu_{312} = (n-2)\nu_{12}$$

(Count triples containing a 12 pattern . . .)

Sketch of Proof: Patterns in Av(123)

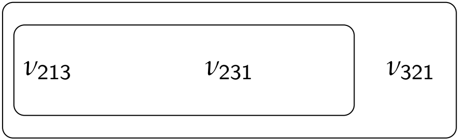
ν_{132} ν_{213} ν_{231} ν_{312} ν_{321}

Sketch of Proof: Patterns in Av(123)

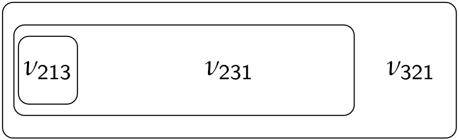
$$\nu_{132} = \nu_{213} \quad \nu_{231} = \nu_{312} \quad \nu_{321}$$

(Since Av(123) is closed under inversion)

Sketch of Proof: Patterns in Av(123)



Sketch of Proof: Patterns in Av(123)



Sketch of Proof: Counting 213 Patterns

Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.

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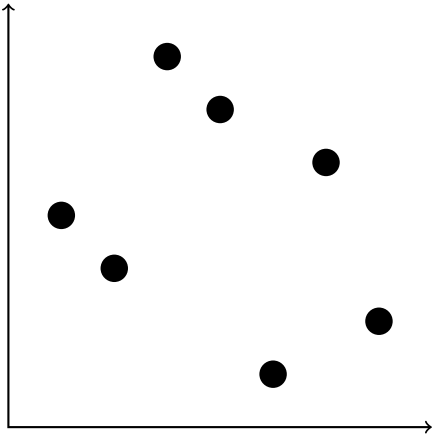

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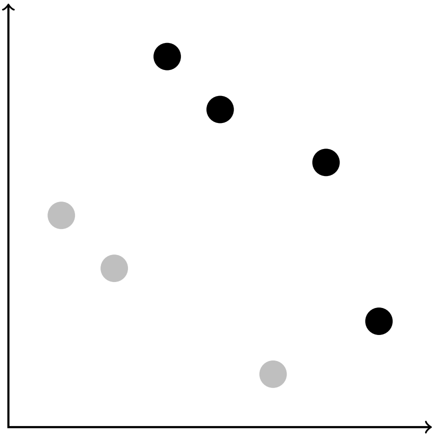

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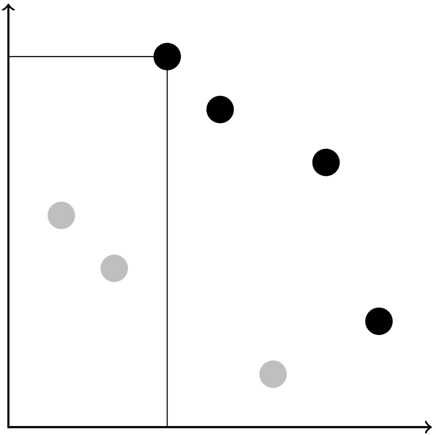

$$\nu_{213}(p) = \binom{2}{2}$$

Sketch of Proof: Counting 213 Patterns

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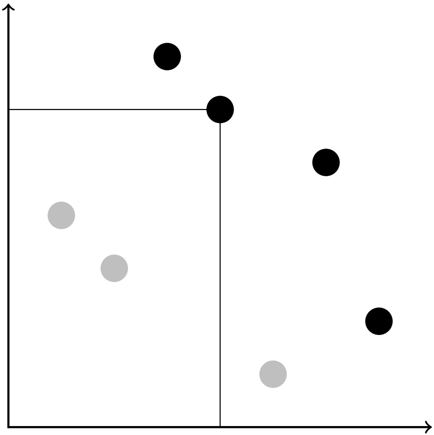

$$\nu_{213}(p) = \binom{2}{2} + \binom{2}{2}$$

Sketch of Proof: Counting 213 Patterns

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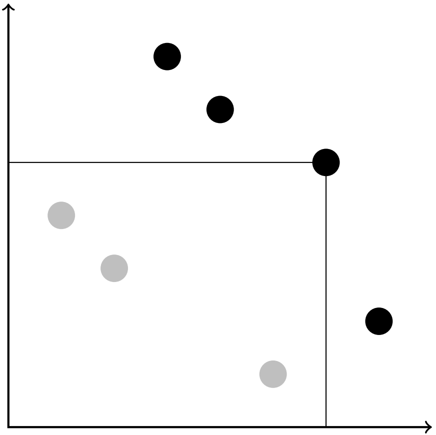

$$\nu_{213}(p) = \binom{2}{2} + \binom{2}{2} + \binom{3}{2}$$

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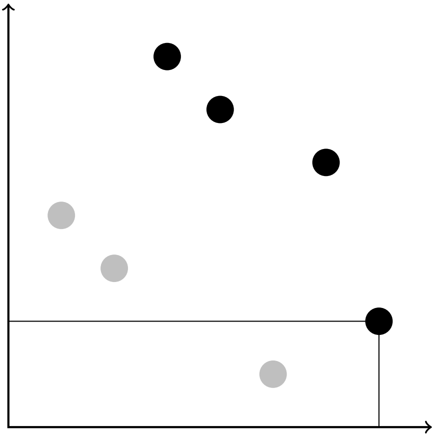

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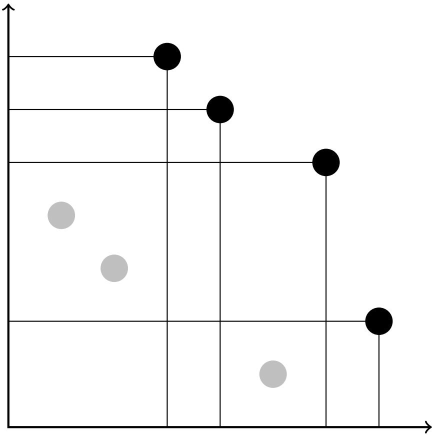

$$\nu_{213}(p) = \binom{2}{2} + \binom{2}{2} + \binom{3}{2} + \binom{1}{2} = 5$$

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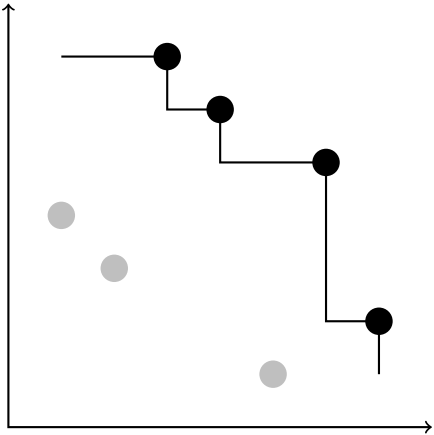

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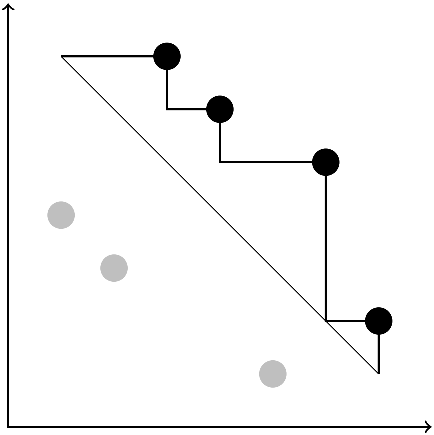

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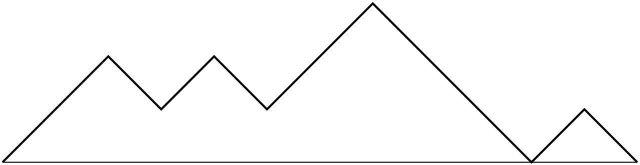

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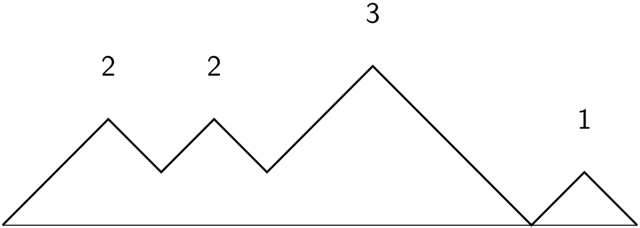

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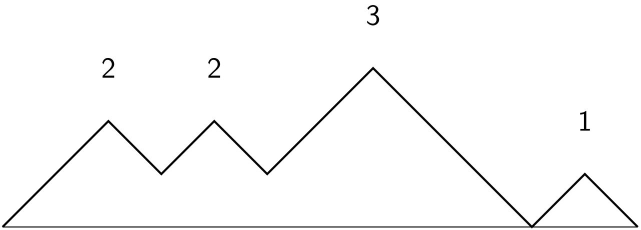
Sketch of Proof: Counting 213 Patterns

Let $h_{n,k}$ denote the total number of peaks at height k in all Dyck paths of semilength n. Let $H(x,u) = \sum_{n,k\geq 0} h_{n,k} x^n u^k$.



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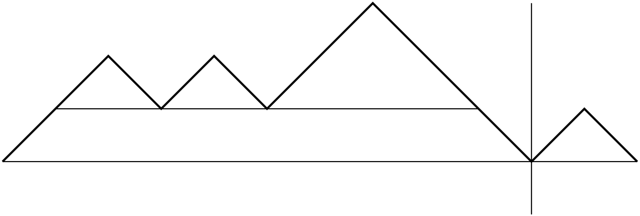

$$H(x, u) =$$

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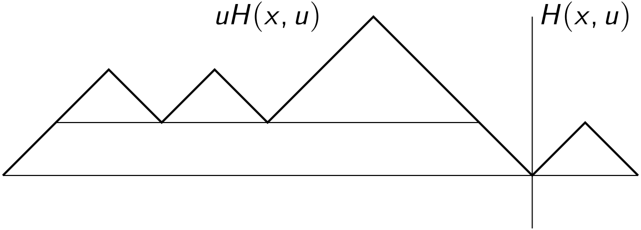

$$H(x,u) = ux(H(x,u)+1)C(x) + xC(x)H(x,u)$$

Sketch of Proof: Counting 213 Patterns

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$$H(x,u) = \frac{uxC(x)}{1 - uxC(x) - xC(x)}.$$

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$$= \frac{x^3 C(x)}{(1-4x)^{3/2}}$$

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$$= x^3 + 7x^4 + 38x^5 + 187^6 + \ldots$$

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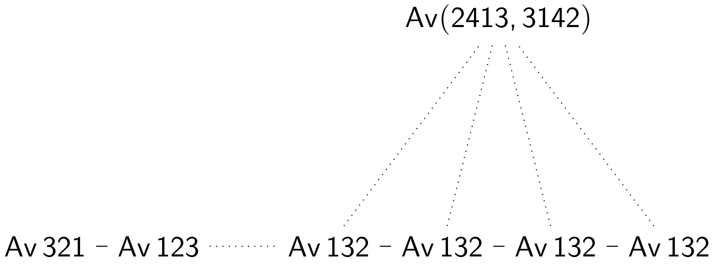
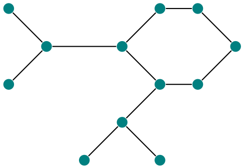
□

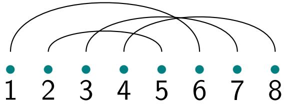
What We Know



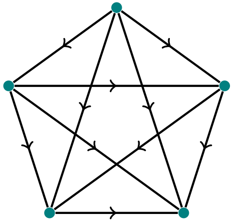
Combinatorial Classes



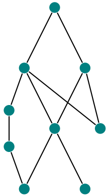
Graphs



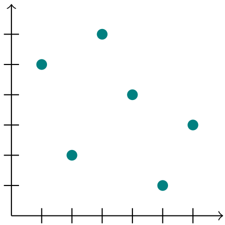
Matchings



Tournaments



Posets



Permutations