# Trees, Paths, and Pocket Change

## An Introduction to Analytic Combinatorics

Cheyne Homberger

March 11, 2013

# Plane Trees

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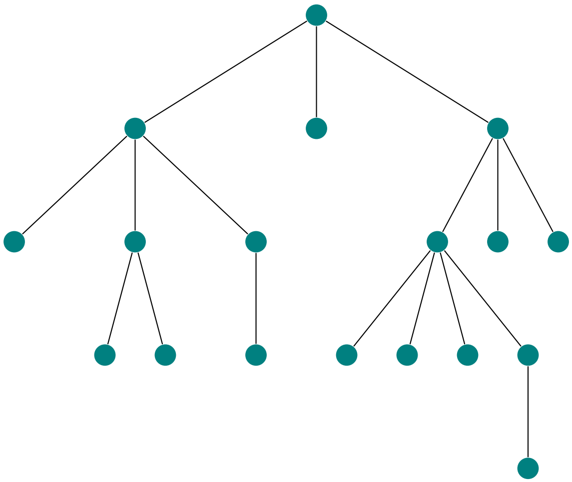

# Plane Trees

1

number of trees

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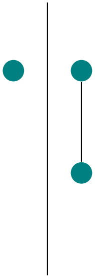

1 1

number of trees

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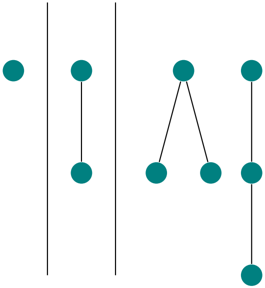

1 1 2

number of trees

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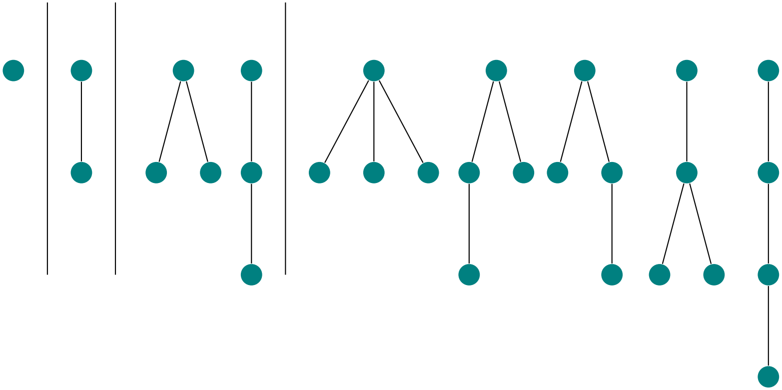

1 1 2 5

number of trees

# Binary Trees

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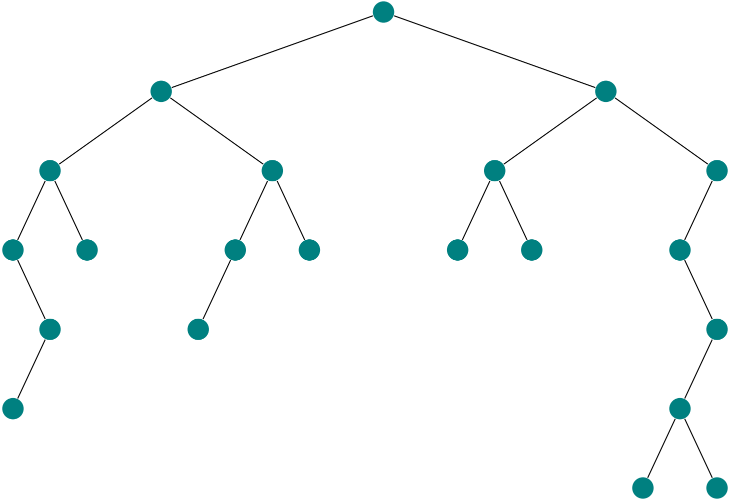

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number of trees

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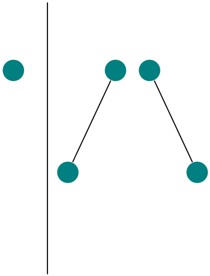

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number of trees

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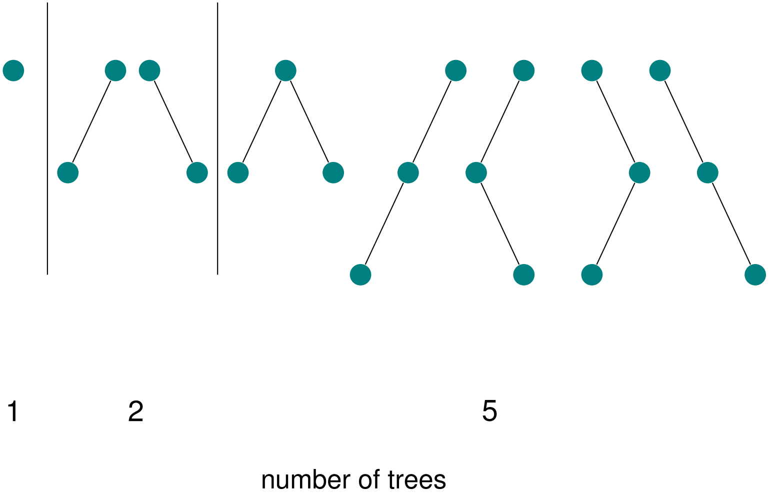

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$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k(k-1)(k-2)\ldots \cdot 3 \cdot 2 \cdot 1}$$

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## Example

$$\binom{5}{1} = 5$$

$$\binom{5}{2} = \frac{5\cdot 4}{2} = 10$$

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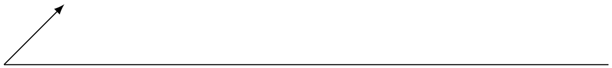

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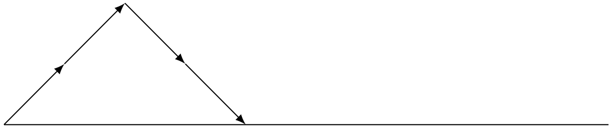

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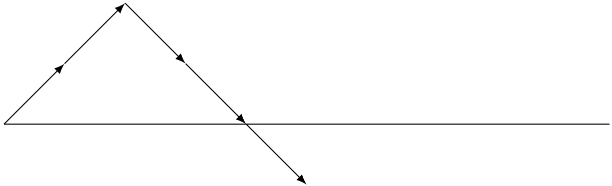

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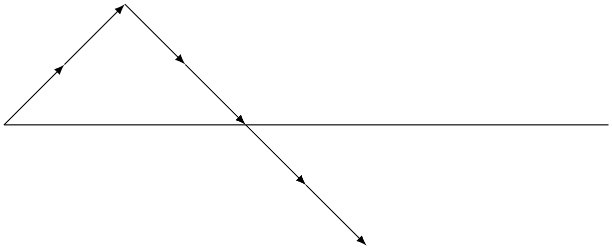

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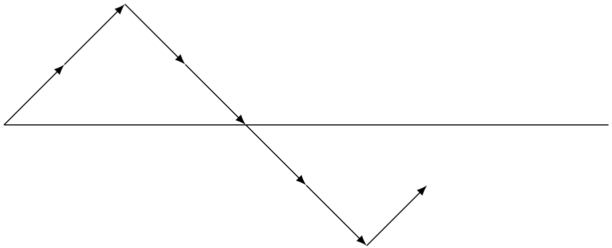

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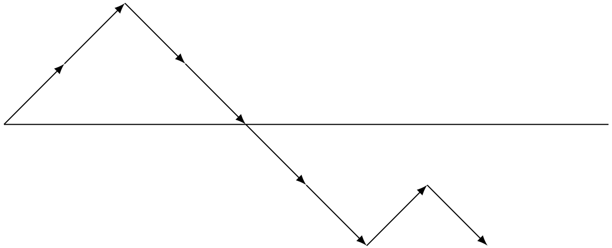

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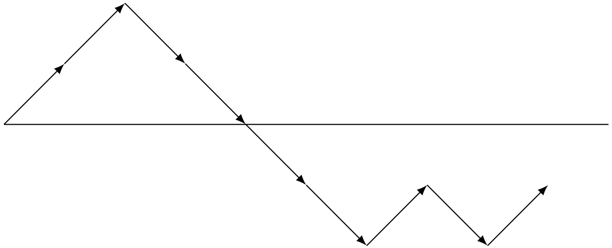

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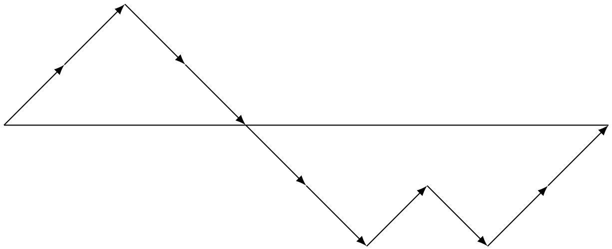

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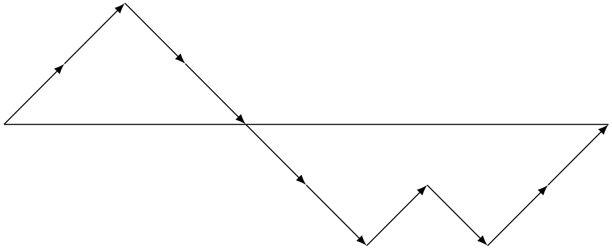

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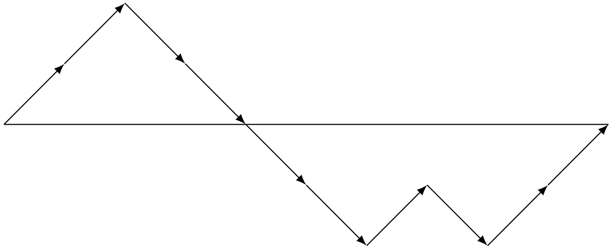

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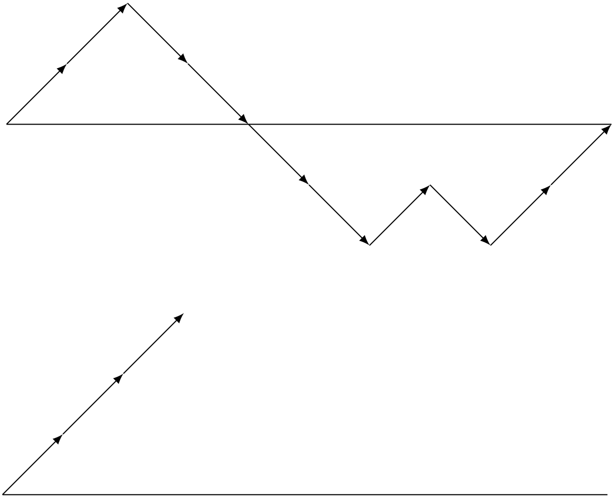

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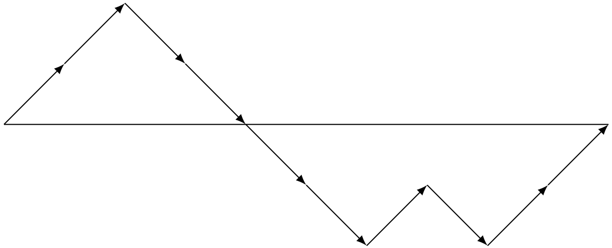

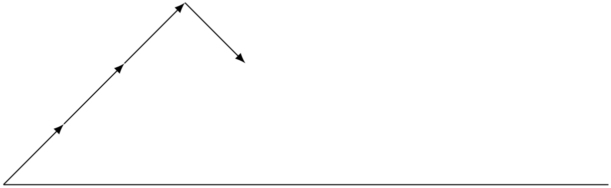

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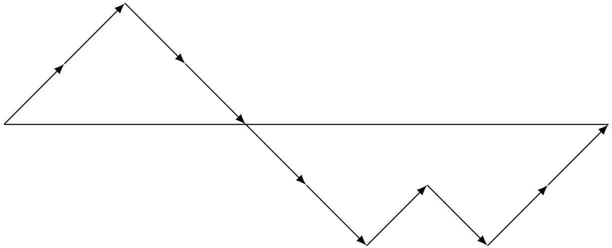

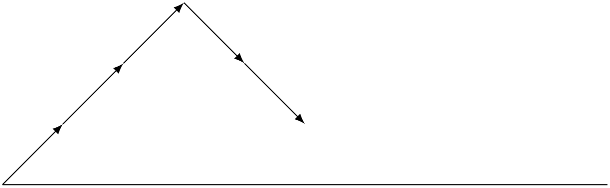

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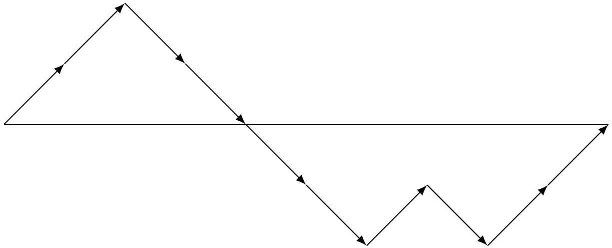

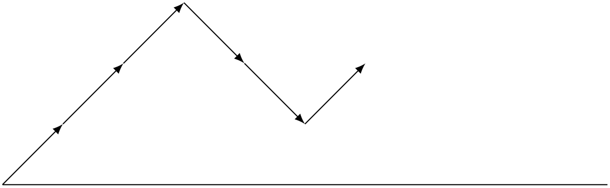

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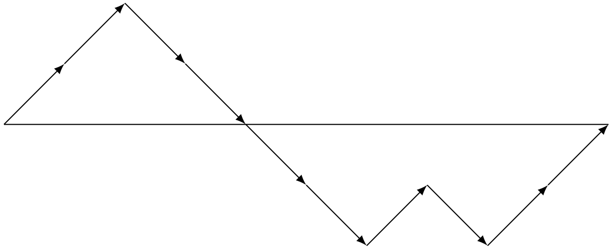

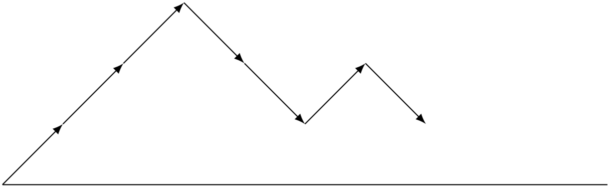

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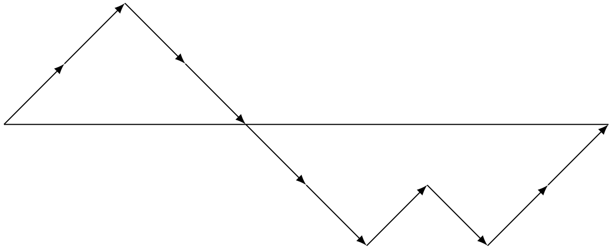

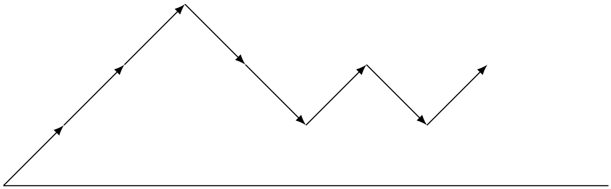

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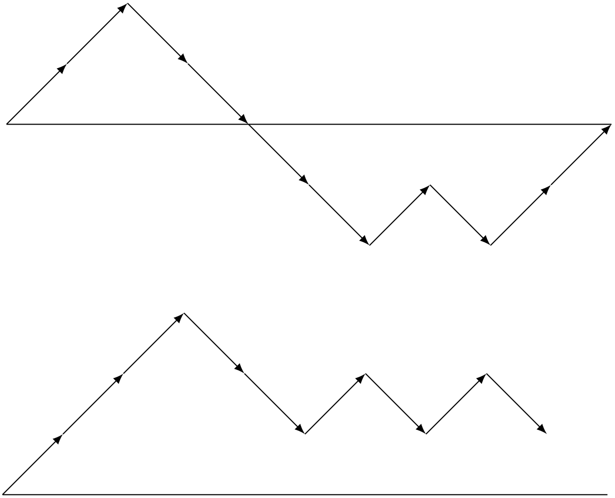

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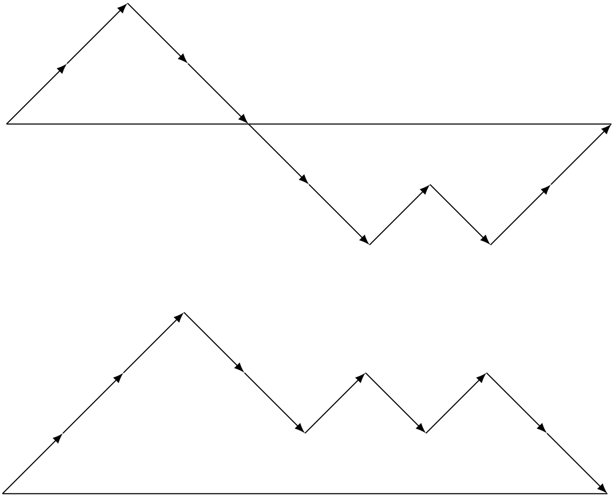

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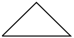

1

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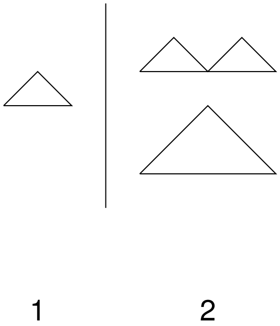

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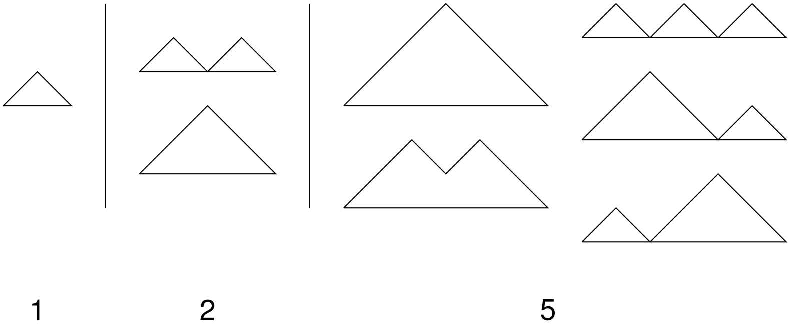

# The Distributive Property

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**Fact**

$$c(a+b) = ca + cb$$

# Binomial Theorem

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## Proof.

$$\underbrace{(x+1)(x+1)(x+1)\dots(x+1)}_{n \text{ terms}} = a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0$$

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$$2^n = \sum_{k=0}^{n} \binom{n}{k} \qquad 0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}.$$

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## Definition

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{n \geq 0} a_n x^n$$

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## Example

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$$= 1+2x+3x^2+4x^3+\dots$$

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Let $F = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \ldots$

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$$\begin{array}{rl} xF = & x + \quad x^2 + \quad 2x^3 + \quad 3x^4 + \quad 5x^5 + 8x^6 + \ldots \\ + x^2F = & \qquad\quad x^2 + \quad x^3 + \quad 2x^4 + \quad 3x^5 + 5x^6 + \ldots \\ \hline \end{array}$$

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$$\begin{array}{rcccccc}
xF = & & x+ & x^2+ & 2x^3+ & 3x^4+ & 5x^5 + 8x^6 + \ldots \\
+\,x^2F = & & & x^2+ & x^3+ & 2x^4+ & 3x^5 + 5x^6 + \ldots \\
\hline
F = & 1+ & x+ & 2x^2+ & 3x^3+ & 5x^4+ & 8x^5 + 13x^6 + \ldots
\end{array}$$

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$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Let $F = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \ldots$

$$\begin{array}{rcccccl}
xF = & & x+ & x^2+ & 2x^3+ & 3x^4+ & 5x^5 + 8x^6 + \ldots \\
+\, x^2F = & & & x^2+ & x^3+ & 2x^4+ & 3x^5 + 5x^6 + \ldots \\
\hline
F = & 1+ & x+ & 2x^2+ & 3x^3+ & 5x^4+ & 8x^5 + 13x^6 + \ldots
\end{array}$$

$$F = xF + x^2F + 1$$

# Fibonacci Numbers

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\end{array}$$

$$F = xF + x^2F + 1$$

$$F - xF - x^2F = 1$$

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$$\begin{array}{rcccccl} xF = & & x+ & x^2+ & 2x^3+ & 3x^4+ & 5x^5 + 8x^6 + \ldots \\ +x^2F = & & & x^2+ & x^3+ & 2x^4+ & 3x^5 + 5x^6 + \ldots \\ \hline F = & 1+ & x+ & 2x^2+ & 3x^3+ & 5x^4+ & 8x^5 + 13x^6 + \ldots \end{array}$$

$$F = xF + x^2F + 1$$

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$$(1 - x - x^2)F = 1 \rightarrow F = \frac{1}{1 - x - x^2}$$

# Fibonacci Numbers Cont'd

$$F = \frac{1}{1 - x - x^2}$$

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$$= \frac{-1}{(x - \alpha)(x - \beta)}$$

# Fibonacci Numbers Cont'd

$$
\begin{aligned}
F &= \frac{1}{1 - x - x^2} \\
&= \frac{-1}{(x-\alpha)(x-\beta)} \\
&= \frac{A}{x-\alpha} + \frac{B}{x-\beta}
\end{aligned}
\qquad
\alpha = \frac{-1+\sqrt{5}}{2}, \beta = \frac{-1-\sqrt{5}}{2}
$$

# Fibonacci Numbers Cont'd

$$\begin{aligned} F &= \frac{1}{1-x-x^2} \qquad \alpha = \frac{-1+\sqrt{5}}{2}, \beta = \frac{-1-\sqrt{5}}{2} \\ &= \frac{-1}{(x-\alpha)(x-\beta)} \\ &= \frac{A}{x-\alpha} + \frac{B}{x-\beta} \\ &= \frac{-1/\sqrt{5}}{x-\alpha} - \frac{1/\sqrt{5}}{1-\beta} \end{aligned}$$

# Fibonacci Numbers Cont'd

$$
\begin{aligned}
F &= \frac{1}{1-x-x^2} \qquad \alpha = \frac{-1+\sqrt{5}}{2}, \beta = \frac{-1-\sqrt{5}}{2} \\
&= \frac{-1}{(x-\alpha)(x-\beta)} \\
&= \frac{A}{x-\alpha} + \frac{B}{x-\beta} \\
&= \frac{-1/\sqrt{5}}{x-\alpha} - \frac{1/\sqrt{5}}{1-\beta} \\
&= \frac{\alpha/\sqrt{5}}{1-x/\alpha} - \frac{\beta/\sqrt{5}}{1-x/\beta}
\end{aligned}
$$

# Fibonacci Numbers Cont'd

$$
\begin{aligned}
F &= \frac{1}{1-x-x^2} \qquad \alpha = \frac{-1+\sqrt{5}}{2}, \beta = \frac{-1-\sqrt{5}}{2} \\
&= \frac{-1}{(x-\alpha)(x-\beta)} \\
&= \frac{A}{x-\alpha} + \frac{B}{x-\beta} \\
&= \frac{-1/\sqrt{5}}{x-\alpha} - \frac{1/\sqrt{5}}{1-\beta} \\
&= \frac{\alpha/\sqrt{5}}{1-x/\alpha} - \frac{\beta/\sqrt{5}}{1-x/\beta} \\
&\cdots \\
&= \sum_{n\geq 0} \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}} x^n
\end{aligned}
$$

# Fibonacci Numbers Cont'd

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F &= \frac{1}{1-x-x^2} \qquad \alpha = \frac{-1+\sqrt{5}}{2}, \beta = \frac{-1-\sqrt{5}}{2} \\
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&= \frac{-1/\sqrt{5}}{x-\alpha} - \frac{1/\sqrt{5}}{1-\beta} \\
&= \frac{\alpha/\sqrt{5}}{1-x/\alpha} - \frac{\beta/\sqrt{5}}{1-x/\beta} \\
&\dots \\
&= \sum_{n\geq 0} \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}} x^n = f_0 + f_1 x + f_2 x^2 + \dots
\end{aligned}$$

# Making Change

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$$P = 1 + x + x^{2} + x^{3} + \ldots = \frac{1}{1 - x}$$

What happens when we combine these?

# Making Change

$$QDNP = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$$

$$= (1 + x^{25} + \ldots)(1 + x^{10} + \ldots)(1 + x^5 + \ldots)(1 + x + \ldots)$$

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$$1 + a_1 x + a_2 x^2 + \ldots + a_{12} x^{12} + \ldots$$

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$$1 + a_1 x + a_2 x^2 + \ldots + a_{12} x^{12} + \ldots$$

$$= 1 + x + x^2 + x^3 + x^4 + 2x^5 + 2x^6 + 2x^7 + 2x^8 + 2x^9$$
$$+4x^{10} + 4x^{11} + 4x^{12} + 4x^{13} + 4x^{14} + 6x^{15} + 6x^{16} + \ldots$$

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$$+4x^{10} + 4x^{11} + 4x^{12} + 4x^{13} + 4x^{14} + 6x^{15} + 6x^{16} + \ldots$$

What about no dimes?

$$QNP = 1 + x + x^2 + x^3 + x^4 + x^5 + 2x^6 + 2x^7 + 2x^8 + 2x^9$$

$$+3x^{10} + 3x^{11} + 3x^{12} + 3x^{13} + 3x^{14} + 4x^{15} + 4x^{16} + \ldots$$

# Fixed Number of Coins

What if we also want to keep track of the total number of coins?

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$$(1 + ux + u^2x^2 + \ldots) \cdot (1 + ux^5 + u^2x^{10} + \ldots)$$

$$\cdot(1 + ux^{10} + u^2x^{20} + \ldots) \cdot (1 + ux^{25} + u^2x^{50} + \ldots)$$

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$$= \frac{1}{(1-ux)(1-ux^5)(1-ux^{10})(1-ux^{25})} = G(x,u)$$

$$G(x,u) = \sum_{n,k \geq 0} a_{n,k} u^k x^n$$

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$$G(x, u) = \sum_{n,k \geq 0} a_{n,k} u^k x^n = \sum_{n \geq 0} x^n \left( \sum_{k \geq 0} a_{n,k} u^k \right)$$

$$= \sum_{k \geq 0} u^k \left( \sum_{n \geq 0} a_{n,k} x^n \right)$$

# Fixed Number of Coins

”I have two coins adding up to 35 cents, and *neither* coin is a dime”

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”I have two coins adding up to 35 cents, and *neither* coin is a dime”

”No you don’t”

# Integer Partitions

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## Theorem

$$\sum_{n \geq 0} p(n)x^n = (1 + x + \ldots)(1 + x^2 + \ldots)(1 + x^3 + \ldots)\ldots$$

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$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + \ldots$$

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## Definition

Let $p_d(n)$ be the number of partitions of $n$ into *distinct* pieces.

## Example

$p_d(5) = 3$, since

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### Example

$p_o(5) = 3$, since

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# Restricted Partitions

### Theorem

$p_d(n) = p_o(n)$ for all positive integers $n$.

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$$\sum_{n \geq 0} p_o x^n = (1 + x + x^2 + \ldots)(1 + x^3 + x^6 + \ldots)\ldots$$

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$$= \frac{(1-x^2)(1-x^4)(1-x^6)\ldots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\ldots}$$

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$$= \frac{1}{(1-x)(1-x^3)(1-x^5)\dots}$$
$$= \frac{(1-x^2)(1-x^4)(1-x^6)\dots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\dots}$$
$$= \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\dots$$

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&= \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\dots \\
&= (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\dots
\end{aligned}$$

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&= (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\dots \\
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$$

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$\sqrt{2} \cdot (4n + 1)$ is the *subexponential* part

$3^n$ is the *exponential* part

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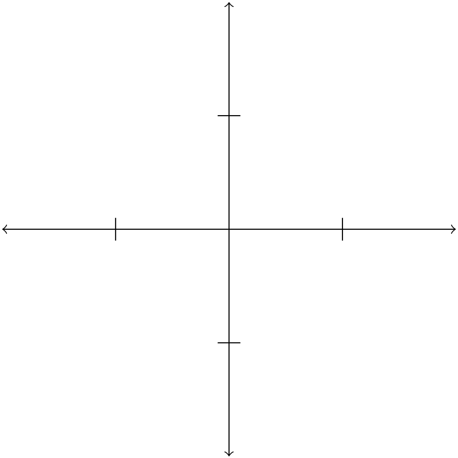

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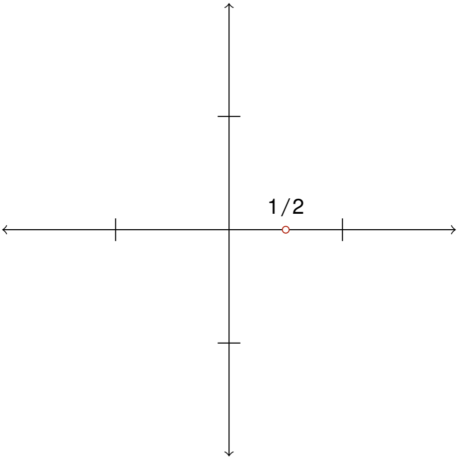

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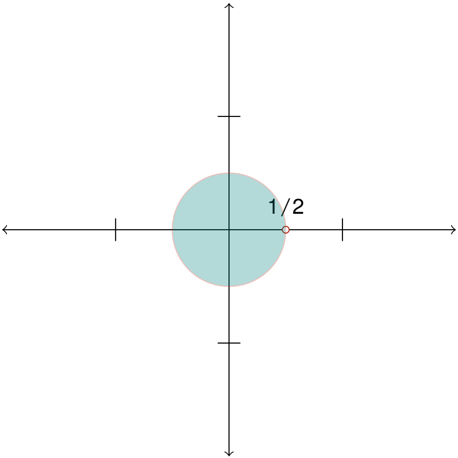

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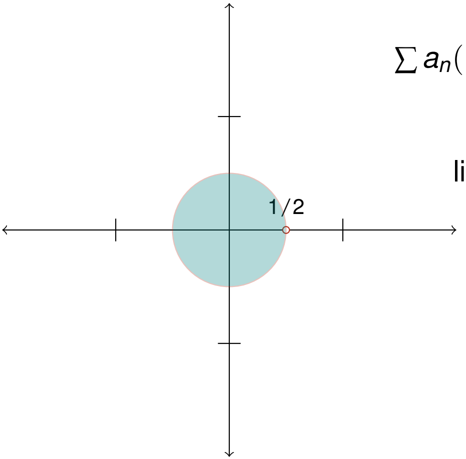


$\sum a_n(1/2)^n$ *barely* diverges

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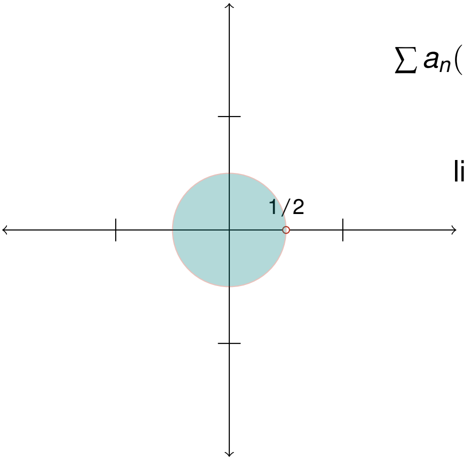


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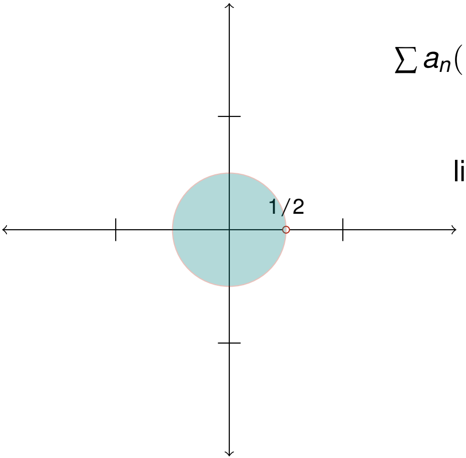


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$a_n = 2^n$

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$$\sum_{n\geq 0} f_n x^n = \frac{1}{1-x-x^2} = \frac{1}{(x-\alpha)(x-\beta)}, \alpha = \frac{-1+\sqrt{5}}{2} \beta = \frac{-1-\sqrt{5}}{2}$$

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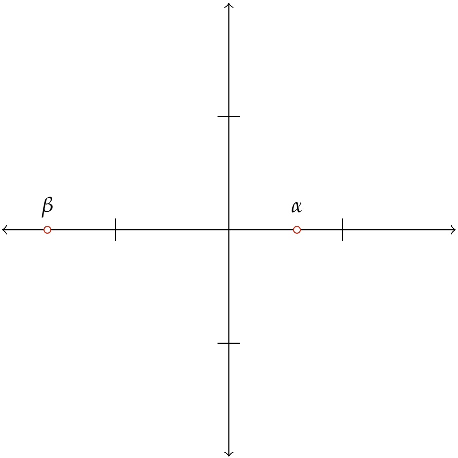

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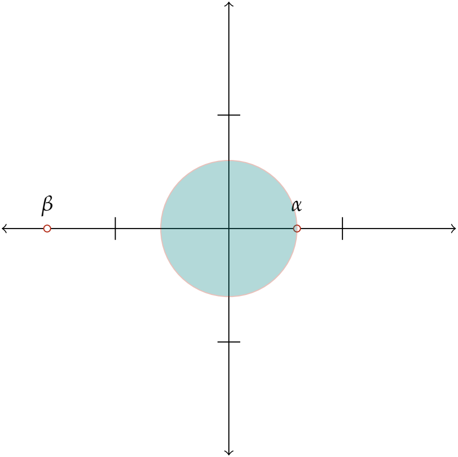
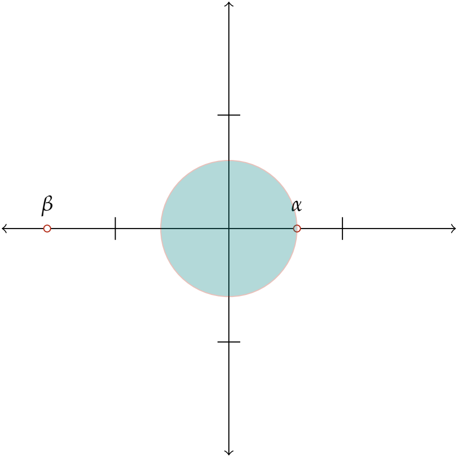


# Fibonacci Asymptotics

$$\sum_{n\geq 0} f_n x^n = \frac{1}{1-x-x^2} = \frac{1}{(x-\alpha)(x-\beta)}, \alpha = \frac{-1+\sqrt{5}}{2} \beta = \frac{-1-\sqrt{5}}{2}$$

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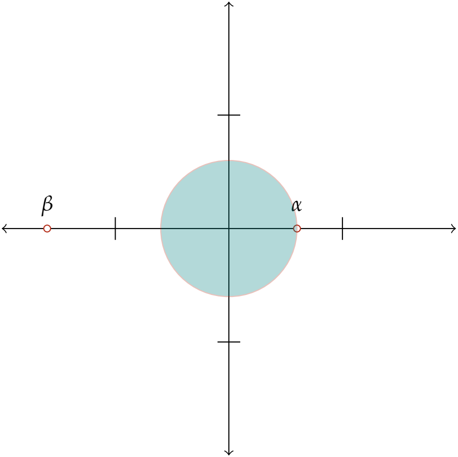


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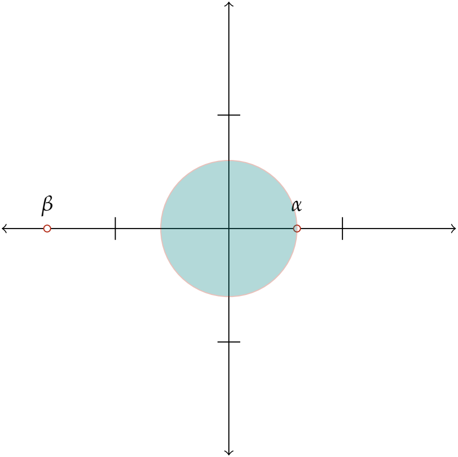


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# Making change (asymptotically)

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Without dimes?

$$\frac{n^2}{250}$$

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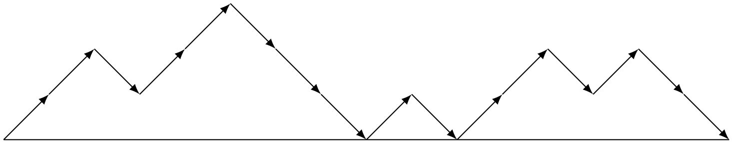

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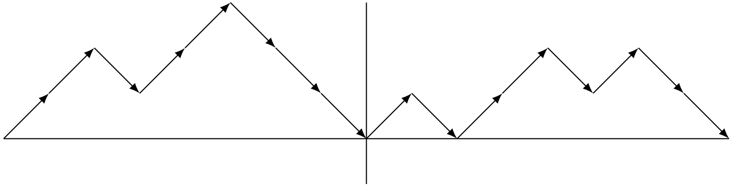

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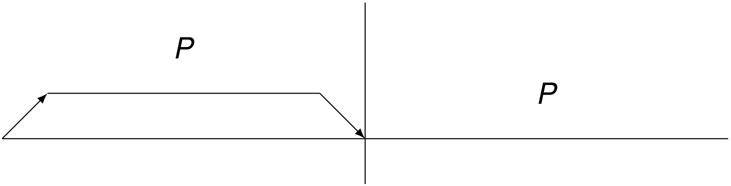

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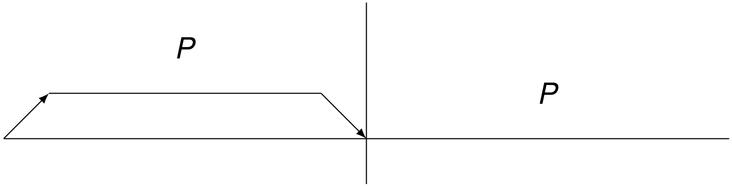


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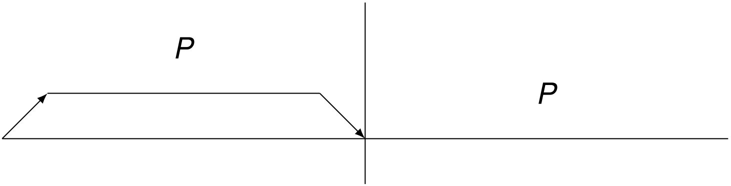


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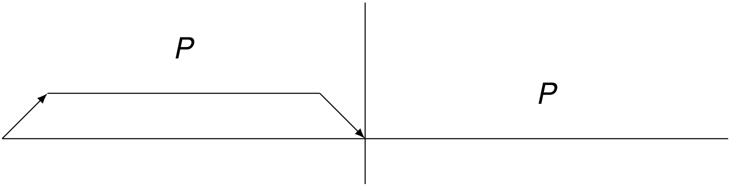


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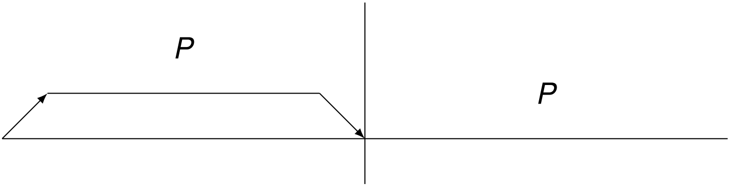


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$$P = \frac{1 - \sqrt{1 - 4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + \ldots$$

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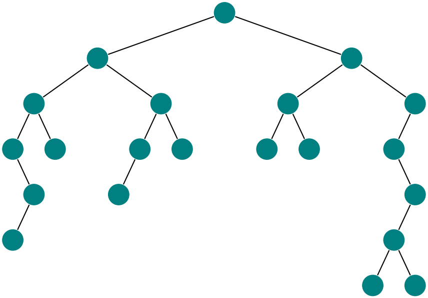

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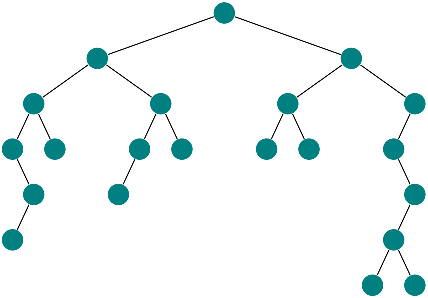

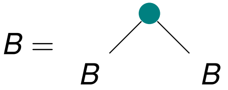


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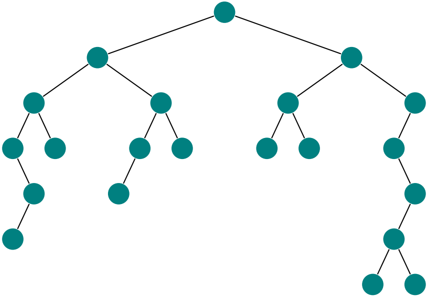

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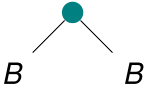


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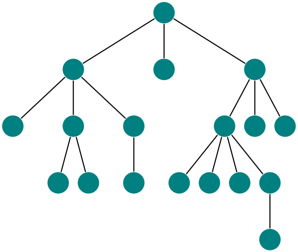

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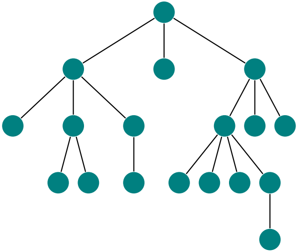

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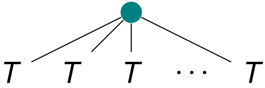

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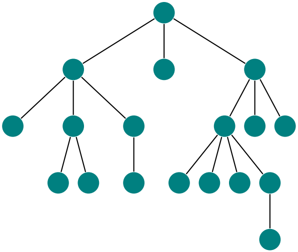

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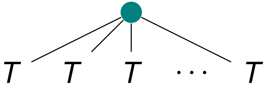


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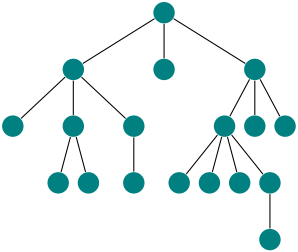

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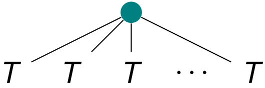


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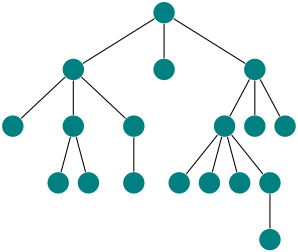

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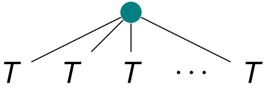


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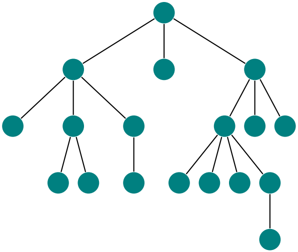

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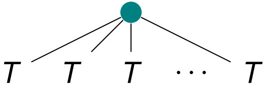


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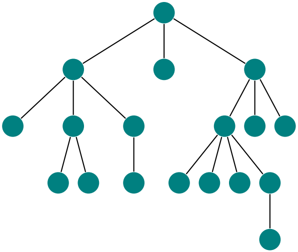

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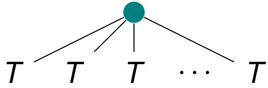


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$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$