Pattern Enumeration in the Separable Permutations

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Joint work with Michael Albert and Jay Pantone

Plotting Permutations

Definition

If π is a permutation of length n, then the *plot* of π is the set of points

$$\{(1, \pi(1)), (2, \pi(2)), \cdots (n, \pi(n))\} \subset \mathbb{R}^2$$

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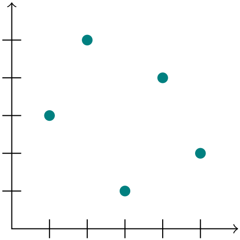

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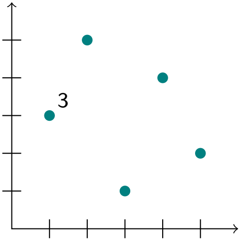

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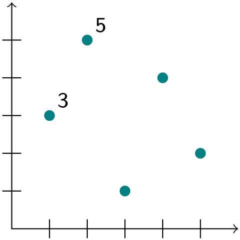

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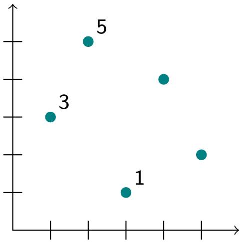

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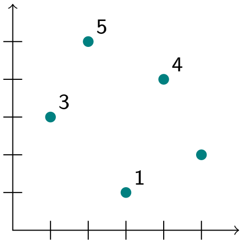

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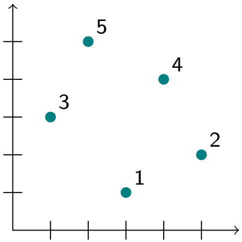

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Dots on a Plane

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Let A and B be two sets of n points in $\mathbb{R}^2$, each with the property that no two points lie on the same horizontal or vertical line.
Say that A is *order isomorphic* to B (denoted $A \sim B$) if A can be transformed into B by stretching, contracting, and translating the axes horizontally and vertically.

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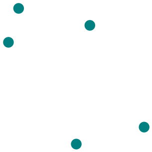
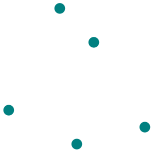


$\sim$



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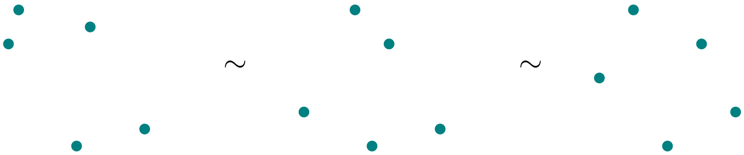

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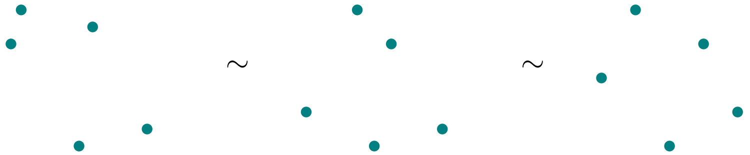

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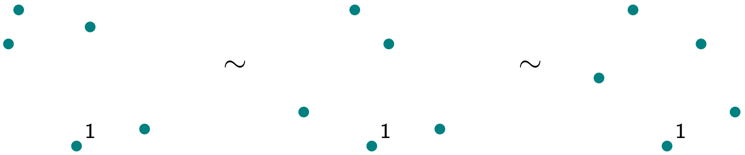

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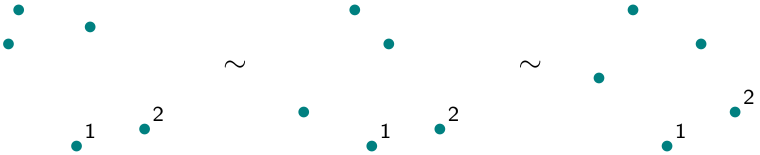

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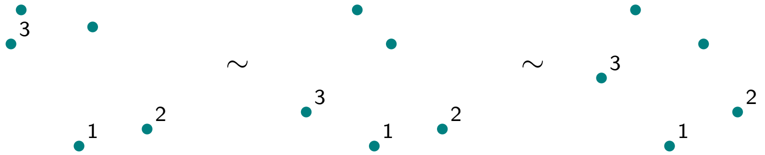

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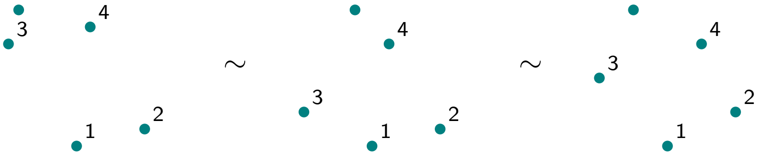

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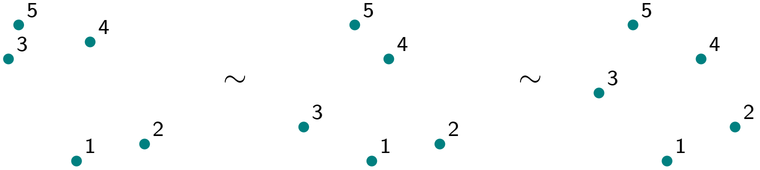

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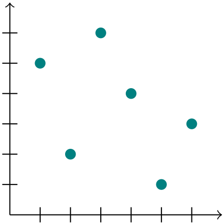

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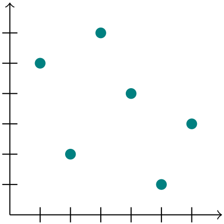

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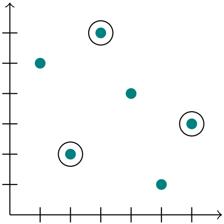

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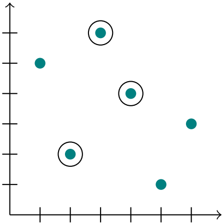

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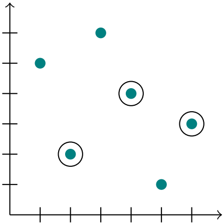

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For a randomly selected permutation of length n, the random variables ν_σ are asymptotically normal as n approaches infinity.

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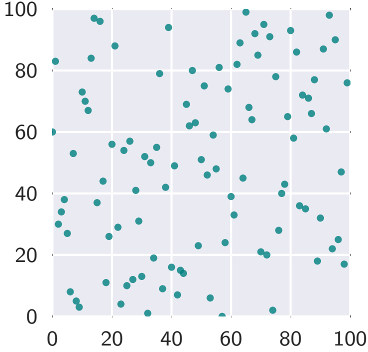
Theorem (Janson, Nakamura, Zeilberger 2013)

For a randomly selected permutation of length n and two patterns σ and ρ, the random variables ν_σ and ν_ρ are asymptotically jointly normally distributed as $n \to \infty$.

Random Permutations



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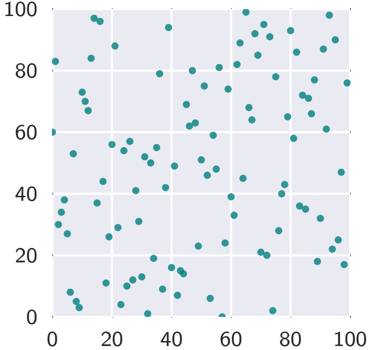

ν_{12}	ν_{21}	Avg
2803	2147	2475

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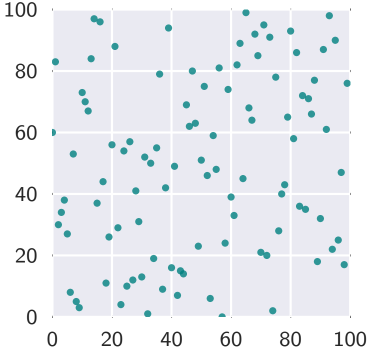

ν_{12}	ν_{21}	Avg
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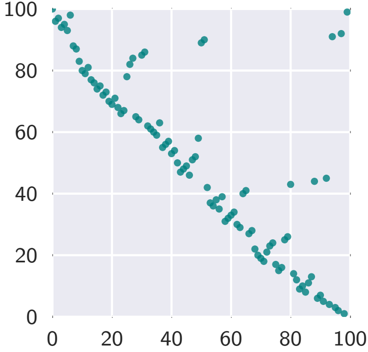
ν_{123}	ν_{132}	ν_{213}	ν_{231}	ν_{312}	ν_{321}	Avg
35357	30063	31414	22321	23348	19197	26950

Random Restricted Permutations



ν_{12}	ν_{21}	Avg
685	4265	2475

ν_{123}	ν_{132}	ν_{213}	ν_{231}	ν_{312}	ν_{321}	Avg
2426	0	14874	15208	14896	114296	26950

Non-Asymptotic Behavior

Fact

In $\mathfrak{S}_n$, the number of occurrences of a specific pattern depends only on the length of the pattern. That is, for a pattern $\sigma \in \mathfrak{S}_k$, we have

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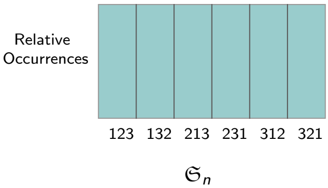

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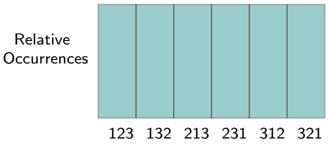
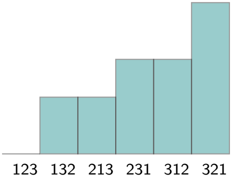


$\mathfrak{S}_n$



$\mathrm{Av}_n\,123$

Equipopularity

Definition

The *popularity* of a pattern σ in a class C is equal to

$$P_\sigma(z) := \sum_{n \geq 1} \nu_\sigma(C_n) z^n.$$

Definition

Patterns are said to be *equipopular* if they have the same number of occurrences (within a specified set or across two different sets).

Equipopularity — Warm up

Fact

For a class C and a pattern σ, we have

$$\nu_\sigma(C_n) = |\{(\pi;\sigma) : \pi \in C_n,\ \sigma \prec \pi\}|.$$

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Proposition

In the class $\mathrm{Av}(132)$, σ and σ^{-1} are equipopular.

Proof.

$$(\pi,\sigma) \mapsto (\pi^{-1},\sigma^{-1}).$$

□

History

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Theorem (Bóna 2010)

Within the class Av(132):

$$\nu_{123} \quad < \quad \nu_{213} = \nu_{231} = \nu_{312} \quad < \quad \nu_{321}.$$

Theorem (H 2012)

$$\nu_{231}(\mathsf{Av}\,132) \;=\; \nu_{231}(\mathsf{Av}\,123).$$

Theorem (Rudolph 2013)

If two patterns *have the same structure*, then they are equipopular within Av(132).

Theorem (Chua, Sankar 2013)

If two patterns are equipopular in Av(132), then they *have the same structure*.

History (in Pictures)

Separable Permutations

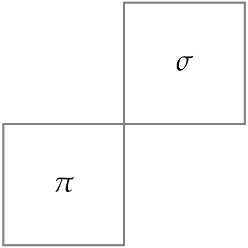
Theorem (Albert, H, Pantone)

Two patterns are equipopular in the separables if and only if they *have the same structure*. Further, the equipopularity classes are in bijection with the set of integer partitions.

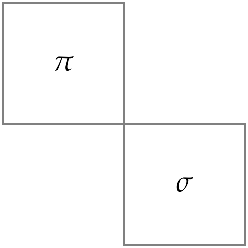
Separable Permutations

Definition

Given two permutations π and σ, their *direct sum* ($\pi \oplus \sigma$) and *skew sum* ($\pi \ominus \sigma$) are defined as follows:



$\pi \oplus \sigma$



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Example

The permutation $\pi = 215643798$ is separable, since

$$\pi = \Big(1 \ominus 1\Big) \oplus \Big((1 \oplus 1) \ominus 1 \ominus 1\Big) \oplus 1 \oplus \Big(1 \ominus 1\Big).$$

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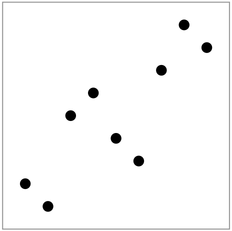

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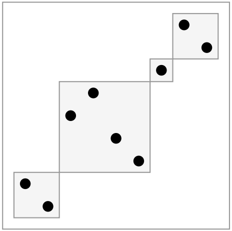

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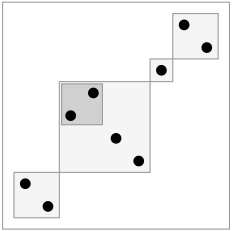

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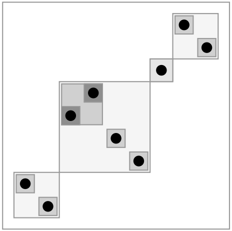

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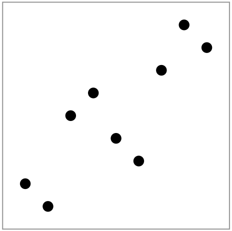

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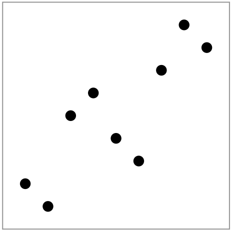

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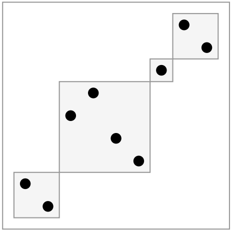

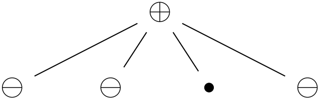

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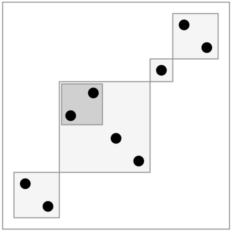

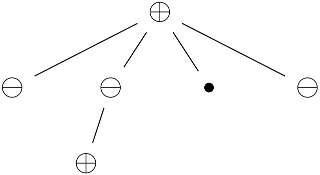

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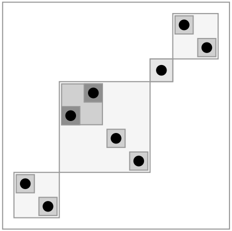
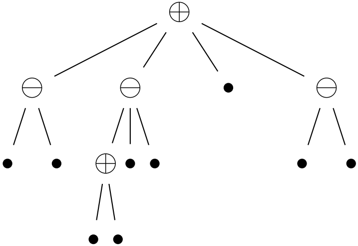
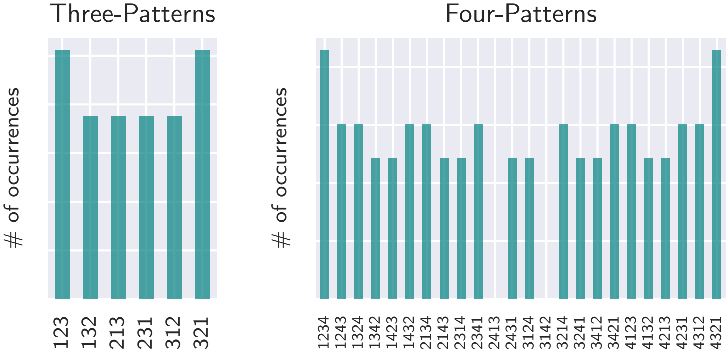




Popularity in the Separables



Tree Patterns

Question

If two patterns are equipopular, how are their trees related?

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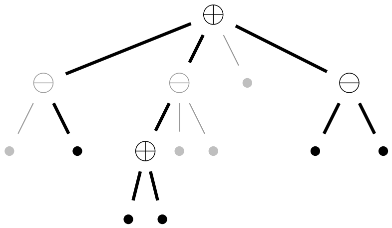
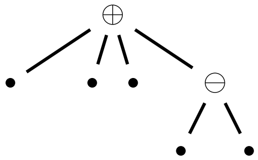
Pattern Containment:





Strategy

Part 1

Find the operations on trees which preserve popularity.

Part 2

Show that equipopularity implies that their trees are related by one of these operations.

Preserving Popularity

Symmetries

Permutation	Tree
Complement	Flip signs
Reverse	Reversal and sign flip
Inverse	Reverse children of $\ominus$ nodes

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Fact

If two permutations (or trees) are related by any of the above symmetries, then they are equipopular.

Proof.

Consider marked patterns. □

Preserving Popularity - Shuffling

Lemma

Rearranging the children of any node preserves equipopularity.

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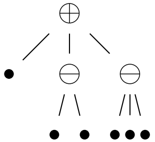

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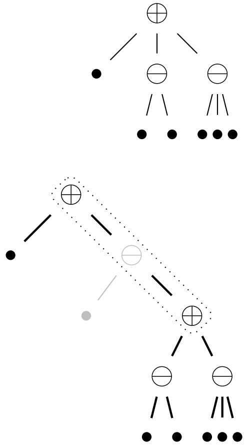

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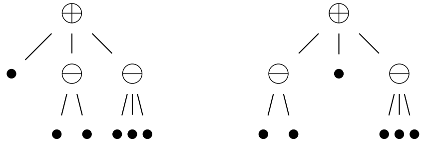

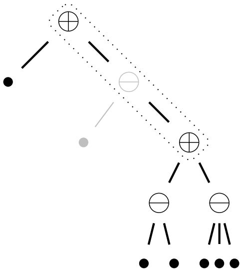

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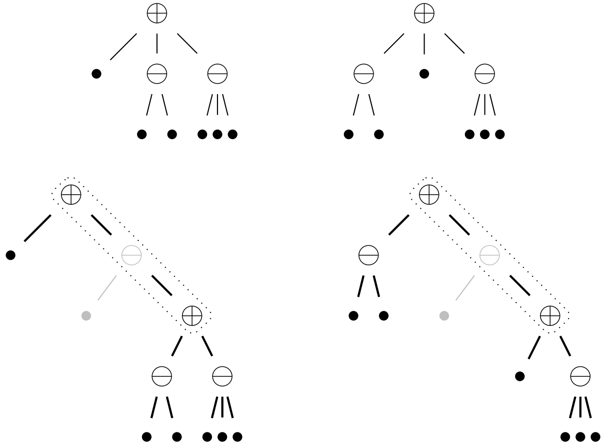

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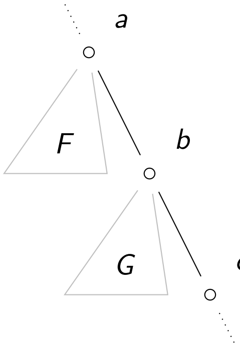

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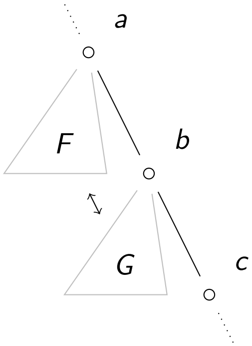

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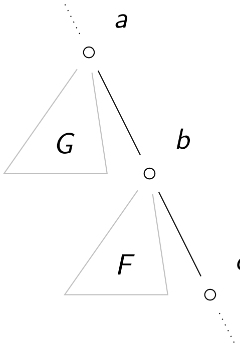


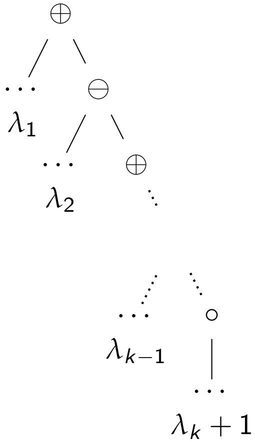
Preserving Popularity

Lemma

The following operations preserve popularity:

- Reversal
- Complementation
- Inversion
- Shuffling
- Rotation

Canonical Representatives



$$\lambda := \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

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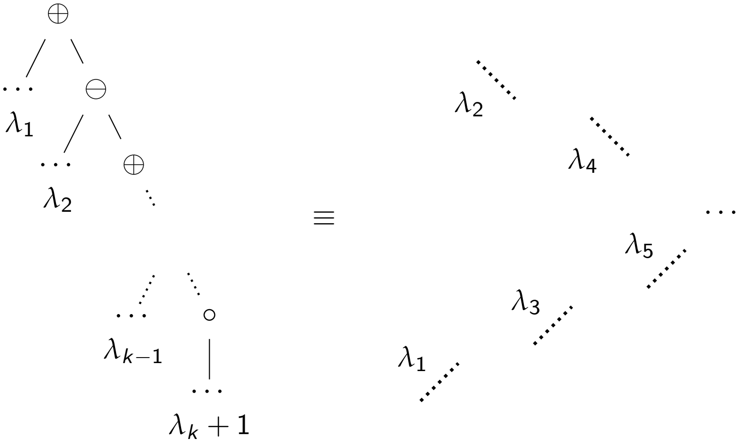

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Lemma

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Corollary

The set of equipopularity classes for patterns of length n are in bijection with the set of partitions of the integer $n-1$.

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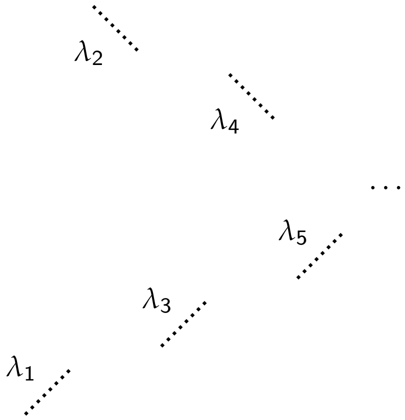

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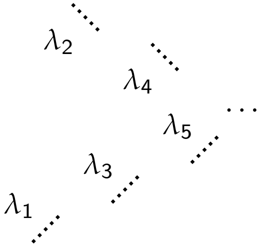
- Notice (or let Sage/Maple/Mathematica/Singular tell you) that these are related to the *Gegenbauer polynomials*, a family of orthogonal polynomials.
- Use the orthogonality of these polynomials to uniquely factor any product.

Conclusion



$$\leftrightarrow \ \lambda_1 + \lambda_2 + \cdots + \lambda_{k-1} + (\lambda_k - 1)$$

Length n canonical representative $\leftrightarrow$ partition of the integer $n - 1$.

Thank You!