PROLIFIC PERMUTATIONS: OPTIMIZING DOWNSETS IN THE PATTERN POSET

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Abstract. A permutation of n letters is k-prolific if each $(n-k)$-subset of the letters in its one-line notation forms a unique pattern. In this paper, we study which values of n can produce such an object. We show that a k-prolific permutation requires at least $k^2/2 + 2k$ letters, and we construct k-prolific permutations using only slightly more than $k^2/2 + 2k$ letters, thus giving a bound for the minimal number of letters needed to form a k-prolific permutation.

Keywords: permutation, pattern, pattern poset, downset, pattern packing, prolific permutation

1. Introduction

The set of permutations of $[n] = \{1, 2, \ldots, n\}$ is denoted S_n, and we write permutations in one-line notation $w = w(1)w(2)\cdots w(n)$. This prompts the classical notion of pattern containment, and we write $\approx$ to denote two strings whose letters appear in the same relative order.

Definition 1.1. Consider $p \in S_k$. A permutation $w \in S_n$ *contains a p-pattern* if there are indices $1 \le i_1 < \cdots < i_k \le n$ such that $w(i_1)\cdots w(i_k) \approx p$. If w contains a p-pattern, we write $p \preceq w$. If w does not contain a p-pattern, then w *avoids* p, denoted $p \not\preceq w$.

From this, it is natural to define the poset of permutation patterns.

Definition 1.2. If $p \preceq w$ but $p \neq w$, then we write $p \prec w$. If $p \prec w$ for $p \in S_k$ and $w \in S_{k+1}$, then w *covers* p. Let $\mathcal{P}$ be the poset whose elements are $S_0 \cup S_1 \cup S_2 \cup \cdots$, with the ordering relation defined by $\preceq$.

This paper is concerned with principal downsets of this poset; that is, the sets of patterns which lie below a given permutation. In particular, we examine those permutations whose downset is as large as possible in the upper ranks. This is in contrast to the related problems of pattern packing [1, 5] or superpatterns [2], each of which seeks to (roughly) maximize the number of relatively small patterns contained in a single permutation. Note also that Smith [6] showed that permutations are uniquely determined by their set of large patterns. The reader is referred to Bóna [3] for an overview of problems related to the permutation pattern poset.

We note that the distribution of prolific permutations (described in Definition 1.5) were previously investigated by the first author in [4]. The main construction presented below

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is a substantial improvement over the results of [4], which were erroneously claimed to be minimal.

We begin with some preliminary facts about downsets. By definition, the elements of rank k in $\mathcal{P}$ are exactly the elements of S_k.

Lemma 1.3. *For a permutation $w \in S_n$, there are at most $\binom{n}{k}$ distinct permutations $p \preceq w$ that lie exactly k ranks below w in $\mathcal{P}$.*

Proof. Permutations less than w in $\mathcal{P}$ are obtained by erasing letters from the one-line notation for w. For each letter erased, the rank decreases by 1. □

Thinking of the covering relations in $\mathcal{P}$ as parent-child relationships, Lemma 1.3 says that a permutation $w \in S_n$ has at most $\binom{n}{1}$ distinct children, at most $\binom{n}{2}$ distinct grandchildren, and so on. The reason for the "at most" in these statements can be seen in the following example.

Example 1.4. The permutation $41253 \in S_5$ covers only $4 < \binom{5}{1}$ elements in $\mathcal{P}$. They are

$$1253 \approx 1243,\ 4253 \approx 4153 \approx 3142,\ 4123,\ \text{and } 4125 \approx 3124.$$

In fact, it is this potential for collapse in the generations below $w \in \mathcal{P}$, as demonstrated in Example 1.4, that piques our interest in this work.

Definition 1.5. Fix positive integers $n > k$. A permutation $w \in S_n$ is *k-prolific* if, for all $i \in [1, k]$,

$$\#\Big\{p \in S_{n-i} : p \preceq w\Big\} = \binom{n}{i}. \tag{1}$$

In fact, as will follow from Lemma 2.5, this definition can be rephrased as follows.

Lemma 1.6. *A permutation $w \in S_n$ is k-prolific if and only if*

$$\#\Big\{p \in S_{n-k} : p \preceq w\Big\} = \binom{n}{k}.$$

It is important to observe that larger values of n do not guarantee more highly prolific permutations.

Example 1.7. For any positive integer n, the identity permutation $12 \cdots n \in S_n$ has only one distinct child in the poset $\mathcal{P}$ of permutation patterns (namely, $12 \cdots (n-1) \in S_{n-1}$). In other words, the identity permutation in S_n is never 1-prolific.

This, and observations like it, suggests the following definition.

Definition 1.8. Given a positive integer k, let $\mathsf{MinProl}(k)$ be the minimum value for which there exists a k-prolific permutation $w \in S_{\mathsf{MinProl}(k)}$.

Example 1.9. $\mathsf{MinProl}(1) = 4$, and the 1-prolific elements of S_4 are 3142 and 2413. The first of these covers the four permutations $\{132, 231, 312, 213\} \subset S_3$, and the second covers $\{312, 213, 132, 231\} \subset S_3$. There are no 1-prolific permutations in S_2 because the permutations 12 and 21 each cover only one element in $\mathcal{P}$. Similarly, permutations in S_3 each cover only one or two elements in $\mathcal{P}$. For example, 123 covers only 12, while 213 covers only 12 and 21.

We can now state the central question of our work.

Question 1.10. What is the function $\mathsf{MinProl}(k)$?

By Definition 1.5, we certainly have

$$\mathsf{MinProl}(k) > k \tag{2}$$

but this is a poor bound. In fact, we can describe the function MinProl much more accurately. In Section 2, we will show that

$$\mathsf{MinProl}(k) \geq k^2/2 + 2k,$$

meaning that there exist k-prolific permutations in S_n only if $n \geq k^2/2 + 2k$. In Section 3 we will construct k-prolific permutations proving that

$$\mathsf{MinProl}(k) \leq \begin{cases} k^2/2 + 3k + 1/2 & \text{if } k \text{ is odd, and} \\ k^2/2 + 4k + 4 & \text{if } k \text{ is even.} \end{cases}.$$

The similarity between these bounds means that we have very good, if not exact, answer to Question 1.10.

In the final section of the paper, we discuss further directions for this work.

2. A lower threshold

In this section, we give a lower threshold to the function MinProl. Our first step toward improving the threshold given in Inequality (2) is to reframe the quality of being k-prolific in terms of the one-line notation for a permutation.

For a given permutation w, the *plot* of w is the collection of points $(w, w(i))$ in the plane $\mathbb{Z}^2$ (see Figure 1). This viewpoint motivates several definitions. For example, we now have a distance function given by the taxicab metric.

Definition 2.1. For $w \in S_n$ and $i, j \in [n]$,

$$d_w(i,j) = |i - j| + |w(i) - w(j)|.$$

The *breadth* of w is the minimum distance between any two entries; that is, the breadth of w is

$$\min_{i,j} d_w(i,j).$$

Example 2.2. For $w = 41253$, we have $d_w(1,3) = 2 + 2 = 4$, while $d_w(1,4) = 3 + 1 = 4$.

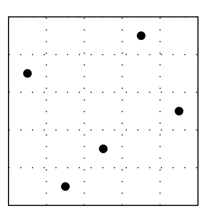

FIGURE 1. The plot of the permutation 41253.

Definition 2.3. Fix $w \in S_n$. For any $i, j \in [n]$, the *span* in w of i and j is the set of entries of w whose positions lie strictly between i and j or whose values lie strictly between $w(i)$ and $w(j)$.

For a permutation $w \in S_n$, let

$$w_{<i>} \in S_{n-1}$$

be the pattern formed by deleting the ith letter from w. Similarly, if $I = \{i_1, i_2, \ldots, i_r\}$, then let $w_{<I>} \in S_{n-r}$ be the pattern formed by deleting the i_1th, i_2th, …, i_rth letters from w.

We start with two lemmas, adapted from [4].

Lemma 2.4. *Deleting an entry from a permutation decreases the breadth by at most one.*

Proof. Deleting an entry decreases the distance between any pair by at most two. Let w be a permutation having breadth m, with i and j such that $d_w(i, j) = |i - j| + |w(i) - w(j)| = m$. The only way to decrease the distance between these entries through deletion is to delete an element of their span. If that deletion were to decrease the distance by two, then we would have deleted a letter whose position k was between i and j and whose value $w(k)$ was between $w(i)$ and $w(j)$. However, this would then imply that the taxicab distance in w between i and k was smaller than $m = d_w(i, j)$, contradicting the minimality of m. □

We use this result to prove the following lemma, which will be instrumental in proving the main theorems of this work.

Lemma 2.5. *A permutation $w \in S_n$ is k-prolific if and only if its breadth is at least $k + 2$.*

Proof. First suppose that $i \neq j$ are such that $|i - j| + |w(i) - w(j)| \leq k + 1$. Let S be the set of elements in the span of i and j. It follows that $|S| \leq k - 1$, and so deleting the sets $S \cup \{i\}$ or $S \cup \{j\}$ results in an identical permutation, meaning that w is not k-prolific.

For the other direction of the theorem, we proceed by induction on k.

Suppose that the breadth of w is at least three, but that w is not 1-prolific; that is, there exists $i \neq j$ such that $w_{<i>} = w_{<j>}$. Assume without loss of generality that $i < j$ and $w(i) < w(j)$. The $(j-1)$st entry in $w_{<i>}$ is $w(j)-1$, while the $(j-1)$st entry in $w_{<j>}$ is $w(j-1)$. But if $w(j)-1 = w(j-1)$, then the breadth of w is at most $|j-(j-1)|+|w(j)-w(j-1)| = 2$, a contradiction. Therefore w must be 1-prolific.

Now fix $k > 1$ and suppose that the breadth of w is at least $k + 2$, but that w is not k-prolific; that is, there are distinct k-element subsets A and B such that $w_{<A>} = w_{<B>}$. If $c \in A \cap B$, then $w' := w_{<c>}$ is $(k - 1)$-prolific, by Lemma 2.4 and induction. However, we must also have $w'_{<A\setminus\{c\}>} = w'_{<B\setminus\{c\}>}$, which would force $A = B$. Thus A and B must be disjoint.

Assume, without loss of generality, that the smallest element $a \in A$ is less than the smallest element $b \in B$, and let $j \in [n]$ be minimal such that $j > a$ but $j \notin A$. Consider the jth entry of $w_{<A>} = w_{<B>}$. This entry is fulfilled by $w(j)$ in $w_{<A>}$ and by $w(a)$ in $w_{<B>}$, which would imply that $|a - j| + |w(a) - w(j)| < k + 2$, a contradiction. □

We can use Lemma 2.5 to give a lower threshold to the function $\mathsf{MinProl}$, substantially improving the one given in Inequality (2).

Theorem 2.6. *For all positive integers k,*

$$\mathsf{MinProl}(k) \geq \frac{k^2}{2} + 2k.$$

Proof. For $w \in S_n$, we can plot the points $\{(i, w(i))\}$ in the square $[1/2, n+1/2]^2$ of area n^2. Lemma 2.5 tells us that if w is k-prolific, then centered at each point $(i, w(i))$ is a diamond of semidiagonal length $k/2+1$, and all of these diamonds are disjoint except perhaps along their boundaries (see Figure 2). The area of each of these n diamonds is

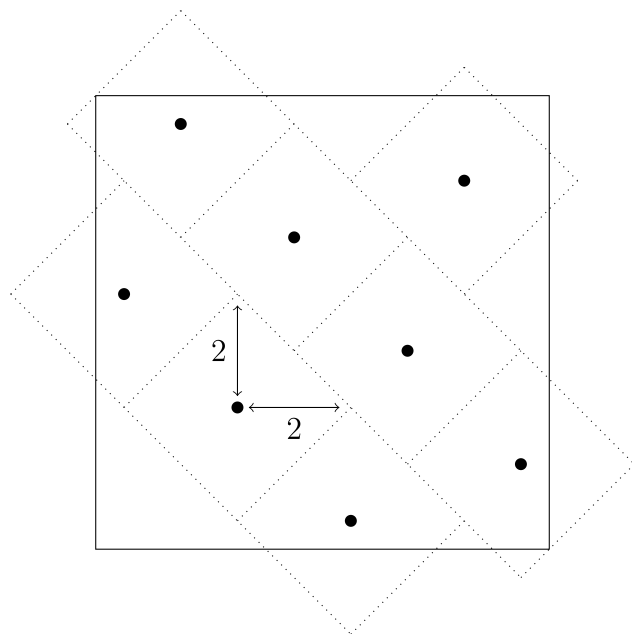

FIGURE 2. The 2-prolific permutation $58361472 \in S_8$ drawn as a grid, with disjoint diamonds of semidiagonal length 2 drawn around each point.

$$2(k/2+1)^2 = (1/2)(k+2)^2 = k^2/2 + 2k + 2,$$

so we would expect to find that $(1/2)(k+2)^2 n$ is roughly equal to n^2. We must say "roughly" because, as demonstrated in Figure 2, there will be some diamonds that extend beyond the bounding square, and there will be some areas of the bounding square that are not covered by diamonds. To be more accurate, we would need to write

$$n^2 = \frac{(k+2)^2}{2} n - (\text{over-coverage}) + (\text{under-coverage}).$$

Consider, without loss of generality, the left edge of the bounding square. The diamonds that extend to the left of this edge are exactly those that are centered at the points $\{(i, w(i)) : i \in [1, (k+1)/2]\}$, and the total area by which this collection of diamonds overflows the boundary square's left edge is

$$\sum_{i \in [1,(k+1)/2]} \left(\frac{k+1}{2} - i\right)^2.$$

The same can be said about the other three sides of the boundary square. Note that if there are any points $(i, w(i))$ that are "close" to a corner of the boundary square (that is, if both $i, w(i) \in [1, (k+1)/2] \cup [n+1-(k+1)/2, n]$), then we will have doubly counted a triangular area of size $(1/2)(k/2+1-i-w(i))^2$. Because we are assuming that w is k-prolific, there is at most one such point for each of the four corners of the boundary square.

Similarly, between any two consecutive diamonds extending beyond the left edge of the boundary square, there is a region inside the boundary square that is not covered by any diamonds. The collection of all $\lfloor (k+1)/2 \rfloor - 1$ such regions has area greater than or equal to

$$\sum_{i \in [2,(k+1)/2]} \left(\frac{k+1}{2} - i \right)^2.$$

Thus, summing over all four sides of the boundary square gives

$$(\text{over-coverage}) - (\text{under-coverage}) \leq 4 \left(\frac{k+1}{2} - 1 \right)^2 = (k-1)^2,$$

so we have

$$\begin{aligned} n^2 &= \frac{(k+2)^2}{2} n - (\text{over-coverage}) + (\text{under-coverage}) \\ &\geq \frac{(k+2)^2}{2} n - (k-1)^2. \end{aligned} \tag{3}$$

It will be helpful to notice that because $k \geq 1$, we have

$$(k+2)^2 - 4 \geq (k-1)^2,$$

so, in particular,

$$-4(k-1)^2 \geq -4(k+2)^2 + 16.$$

We need only consider the positive branch of the quadratic formula applied to Inequality (3), due to Inequality (2). Thus we have

$$\begin{aligned} n &\geq \frac{1}{2} \left[\frac{(k+2)^2}{2} + \sqrt{\left(\frac{(k+2)^2}{2} \right)^2 - 4(k-1)^2} \right] \\ &\geq \frac{1}{2} \left[\frac{(k+2)^2}{2} + \sqrt{\left(\frac{(k+2)^2}{2} \right)^2 - 4(k+2)^2 + 16} \right] \\ &= \frac{1}{2} \left[\frac{(k+2)^2}{2} + \sqrt{\left(\frac{(k+2)^2}{2} - 4 \right)^2} \right] \\ &= \frac{1}{2} \left[\frac{(k+2)^2}{2} + \frac{(k+2)^2}{2} - 4 \right] \\ &= \frac{(k+2)^2}{2} - 2. \end{aligned}$$

Expanding this last expression yields $\mathsf{MinProl}(k) \geq k^2/2 + 2k$, as desired. □

Theorem 2.6 certainly improves upon Inequality (2), but we must note that it is not a perfect bound.

Example 2.7. By Theorem 2.6, we have $\mathsf{MinProl}(1) \geq 2.5$. In fact, as discussed in Example 1.9, $\mathsf{MinProl}(1) = 4$.

3. An upper bound

In this section, we employ a noticeably different technique to give an upper bound to the function $\mathsf{MinProl}(k)$. To do so, we will construct k-prolific permutations in $S_{m(k)}$, where $m(k)$ is a function of k, which implies that $\mathsf{MinProl}(k) \leq m(k)$. The function $m(k)$ is quite similar to the lower threshold given in Theorem 2.6, indicating that this is, in fact, a good bound.

Before describing this function $m(k)$, we introduce a particular permutation defined in terms of a positive integer k.

Definition 3.1. Fix positive integers x and y and let $w_{x,y} \in S_{x(y+1)}$ be the permutation whose one-line notation is as follows, where we use brackets for legibility:

$$1\,\big[1+x\big]\,\big[1+2x\big]\cdots\big[1+xy\big]\,2\,\big[2+x\big]\,\big[2+2x\big]\cdots\big[2+xy\big]\;\;\cdots\;\;x\,\big[2x\big]\,\big[3x\big]\cdots\big[x(y+1)\big].$$

Set

$$z := \max\{x+1, xy\}, \tag{4}$$

and let $\hat{w}_{x,y} \in S_{x(y+1)(z+1)}$ be the permutation obtained by replacing each letter t in $w_{x,y}$ by the string

$$t\,\big[t+x(y+1)\big]\,\big[t+2x(y+1)\big]\,\big[t+3x(y+1)\big]\cdots\big[t+x(y+1)z\big].$$

Note that we can describe the permutation $\hat{w}_{x,y}$ uniquely by

$$\hat{w}_{x,y}(i) = \alpha_i + \beta_i x + \gamma_i x(y+1), \tag{5}$$

where $\alpha_i \in [1,x]$, $\beta_i \in [0,y]$, and $\gamma_i \in [0,z]$ are integers satisfying

$$i = \alpha_i(y+1)(z+1) + \beta_i(z+1) + \gamma_i.$$

Example 3.2.

- If $x = 4$ and $y = 1$, then $z = \max\{5,4\} = 5$ and

$$\hat{w}_{4,1} = 1\ 9\ 17\ 25\ 33\ 41\ 5\ 13\ 21\ 29\ 37\ 45\ 2\ 10\ 18\ 26\ 34\ 42\ 6\ 14\ 22\ 30\ 38\ 46$$
$$3\ 11\ 19\ 27\ 35\ 43\ 7\ 15\ 23\ 31\ 39\ 47\ 4\ 12\ 20\ 28\ 36\ 44\ 8\ 16\ 24\ 32\ 40\ 48 \in S_{48}.$$

- If $x = 1$ and $y = 7$, then $z = \max\{2,7\} = 7$ and

$$\hat{w}_{1,7} = 1\ 9\ 17\ 25\ 33\ 41\ 49\ 57\ 2\ 10\ 18\ 26\ 34\ 42\ 50\ 58\ 3\ 11\ 19\ 27\ 35\ 43\ 51\ 59$$
$$4\ 12\ 20\ 28\ 36\ 44\ 52\ 60\ 5\ 13\ 21\ 29\ 37\ 45\ 53\ 61\ 6\ 14\ 22\ 30\ 38\ 46\ 54\ 62$$
$$7\ 15\ 23\ 31\ 39\ 47\ 55\ 63\ 8\ 16\ 24\ 32\ 40\ 48\ 56\ 64 \in S_{64}.$$

Proposition 3.3. *Fix a positive integer k. If x and y are positive integers such that*

$$x(y+1) = k+1,$$

then the permutation $\hat{w}_{x,y}$ described in Definition 3.1 is k-prolific.

Proof. Thanks to Lemma 2.5, the proof of this statement amounts to checking that

$$|i-j| + |\hat{w}_{x,y}(i) - \hat{w}_{x,y}(j)| \geq k+2$$

for all $i \neq j$. Write $\hat{w}_{x,y}(i)$ and $\hat{w}_{x,y}(j)$ as in Equation (5):

$$\hat{w}_{x,y}(i) = \alpha_i + \beta_i x + \gamma_i x(y+1) \quad \text{and} \quad \hat{w}_{x,y}(j) = \alpha_j + \beta_j x + \gamma_j x(y+1),$$

with $\alpha_i \in [1,x]$, $\beta_i \in [0,y]$, and $\gamma_i \in [0,z]$.

In order to use Lemma 2.5, we translate each of its inequalities into statements about the coefficients $\{\alpha, \beta, \gamma\}$:

$$|i-j| = \Big|(\alpha_i - \alpha_j)(y+1)(z+1) + (\beta_i - \beta_j)(z+1) + (\gamma_i - \gamma_j)\Big|$$

and

$$|\hat{w}_{x,y}(i) - \hat{w}_{x,y}(j)| = \Big|(\alpha_i - \alpha_j) + (\beta_i - \beta_j)x + (\gamma_i - \gamma_j)x(y+1)\Big|.$$

Suppose, without loss of generality, that $i > j$. This means that the string $\alpha_i\beta_i\gamma_i$ is lexicographically greater than $\alpha_j\beta_j\gamma_j$, and $|i-j| = i-j$. We now proceed by cases, using the definition of z given in Equation (4).

- Suppose that $\alpha_i = \alpha_j$. Assume, without loss of generality, that $\beta_j = 0$. Thus

  $$|i-j| = \beta_i(z+1) + (\gamma_i - \gamma_j)$$

  and

  $$|\hat{w}_{x,y}(i) - \hat{w}_{x,y}(j)| = \Big|\beta_i x + (\gamma_i - \gamma_j)x(y+1)\Big|.$$

  - If $\gamma_i \geq \gamma_j$, then

    $$\begin{aligned} |i-j| + |\hat{w}_{x,y}(i) - \hat{w}_{x,y}(j)| &= \beta_i(x+z+1) + (\gamma_i - \gamma_j)(x(y+1)+1) \\ &\geq \beta_i(x+xy+1) + (\gamma_i - \gamma_j)(x(y+1)+1) \\ &= (\beta_i + \gamma_i - \gamma_j)(x(y+1)+1) \\ &\geq x(y+1)+1 \\ &= k+2. \end{aligned}$$

    This is because $i \neq j$ means that $\beta_i \neq 0$ or $\gamma_i \neq \gamma_j$, and so $\beta_i + \gamma_i - \gamma_j \geq 1$.
  - If $\gamma_i < \gamma_j$, then, because $i > j$, we must have $\beta_i > \beta_j = 0$. Thus

    $$\begin{aligned} |i-j| + |\hat{w}_{x,y}(i) - \hat{w}_{x,y}(j)| &= \beta_i(z+1-x) + (\gamma_j - \gamma_i)(x(y+1)-1) \\ &\geq \beta_i(z+1-x) + x(y+1) - 1 \\ &\geq z+1-x+x(y+1)-1 \\ &\geq x+1+1-x+x(y+1)-1 \\ &= x(y+1)+1 \\ &= k+2. \end{aligned}$$

- Now suppose that $\alpha_i > \alpha_j$. Assume, without loss of generality, that $\alpha_j = 1$. Thus we can write

  $$|i-j| = (\alpha_i - 1)(y+1)(z+1) + (\beta_i - \beta_j)(z+1) + (\gamma_i - \gamma_j),$$

and

$$|\hat{w}_{x,y}(i) - \hat{w}_{x,y}(j)| = \Big|(\gamma_i - \gamma_j)x(y+1) + (\beta_i - \beta_j)x + (\alpha_i - 1)\Big|.$$

- If $\alpha_i \geq 3$, then

$$\begin{aligned}|i-j| &= 2(y+1)(z+1) + (\beta_i - \beta_j)(z+1) + (\gamma_i - \gamma_j) \\ &\geq 2(y+1)(z+1) - y(z+1) - z \\ &\geq (y+1)(z+1) + 1 \\ &> x(y+1) + 1 \\ &= k+2.\end{aligned}$$

- If $\alpha_i = 2$, then

$$|i-j| = (y+1+\beta_i - \beta_j)(z+1) + (\gamma_i - \gamma_j),$$

and

$$|\hat{w}_{x,y}(i) - \hat{w}_{x,y}(j)| = \Big|(\gamma_i - \gamma_j)x(y+1) + (\beta_i - \beta_j)x + 1\Big|.$$

  * If $|\gamma_i - \gamma_j| \geq 2$, then, because $i \neq j$, we have

$$\begin{aligned}|i-j| + |\hat{w}_{x,y}(i) - \hat{w}_{x,y}(j)| &\geq 1 + 2x(y+1) - (xy+1) \\ &= 1 + x(y+1) + x - 1 \\ &\geq x(y+1) + 1 \\ &= k+2.\end{aligned}$$

  * If $|\gamma_i - \gamma_j| = 1$, then

$$\begin{aligned}&|i-j| + |\hat{w}_{x,y}(i) - \hat{w}_{x,y}(j)| \\ &\quad\geq (y+1)(z+1) + (\beta_i - \beta_j)(z+1) - 1 + x(y+1) - (\beta_i - \beta_j)x - 1 \\ &\quad\geq (y+1)(x+z+1) + (\beta_i - \beta_j)(-x+z+1) - 2 \\ &\quad\geq (y+1)(x+z+1) + 2(\beta_i - \beta_j) - 2 \\ &\quad\geq x(y+1) + (y+1)(z+1) - 2y - 2 \\ &\quad> x(y+1) + 1 \\ &\quad= k+2.\end{aligned}$$

  * If $\gamma_i = \gamma_j$, then

$$\begin{aligned}|i-j| + |\hat{w}_{x,y}(i) - \hat{w}_{x,y}(j)| &= (y+1+\beta_i - \beta_j)(z+1) + |(\beta_i - \beta_j)x + 1| \\ &\geq (y+1)(z+1) + (\beta_i - \beta_j)(-x+z+1) - 1 \\ &\geq (y+1)(z+1) - y(-x+z+1) - 1 \\ &= xy + z \\ &\geq xy + x + 1 \\ &= x(y+1) + 1 \\ &= k+2.\end{aligned}$$

Thus, by Lemma 2.5, the permutation $\hat{w}_{x,y}$ is k-prolific. □

We now use this construction to give an upper bound for the function $\mathsf{MinProl}$.

Theorem 3.4. *For any positive integer k,*

$$\mathsf{MinProl}(k) \leq \begin{cases} k^2/2 + 3k + 1/2 & \text{if } k \text{ is odd, and} \\ k^2/2 + 4k + 4 & \text{if } k \text{ is even.} \end{cases}$$

Proof. By Proposition 3.3, if x and y are positive integers for which

$$x(y+1) = k+1, \tag{6}$$

then the permutation $\hat{w}_{x,y} \in S_{x(y+1)(z+1)}$ is k-prolific. In fact, the permutation $\hat{w}_{x,y}$ always begins with its smallest entry and ends with its largest entry. Therefore, deleting these two entries to obtain

$$\overline{w}_{x,y} \in S_{x(y+1)(z+1)-2}$$

maintains the breadth of $\hat{w}_{x,y}$ and therefore the fact that the permutation is k-prolific, thus demonstrating that

$$\mathsf{MinProl}(k) \leq x(y+1)(z+1) - 2 = (k+1)(z+1) - 2.$$

Recall the definition of z, given in Equation (4). Because x and y are positive integers, we can restate this as

$$z = \begin{cases} x+1 & \text{if } y = 1, \text{ and} \\ xy & \text{if } y > 1. \end{cases}$$

Equivalently,

$$z = \begin{cases} (k+1-x)+1 & \text{if } y = 1, \text{ and} \\ k+1-x & \text{if } y > 1. \end{cases}$$

The value of k is fixed, so in order to minimize $(k+1)(z+1) - 2$, we must minimize $z+1$. That is, we want to maximize the value of x.

Suppose that k is odd. Then $x = (k+1)/2$ and $y = 1$ are positive integers satisfying Equation (6). Following Equation (4), we have $z = x+1 = (k+3)/2$, and so by constructing $\hat{w}_{x,y}$ and deleting the first and last entries to obtain $\overline{w}_{x,y}$, we have

$$\mathsf{MinProl}(k) \leq (k+1)(z+1) - 2 = (k+1)\left(\frac{k+5}{2}\right) - 2 = \frac{k^2}{2} + 3k + \frac{1}{2}.$$

If k is even, then we cannot factor $k+1$ so conveniently. However, we can use a little slight-of-hand by recalling that any $(k+1)$-prolific permutation is also k-prolific, and so $\mathsf{MinProl}(k) \leq \mathsf{MinProl}(k+1)$. Because $k+1$ is odd, we can argue as above to get that $\mathsf{MinProl}(k+1) \leq (k+1)^2/2 + 3(k+1) + 1/2 = k^2/2 + 4k + 4$. Therefore

$$\mathsf{MinProl}(k) \leq \frac{k^2}{2} + 4k + 4.$$

□

Example 3.5. Theorem 3.4 says that $\mathsf{MinProl}(5) \leq 28$. The plot of the permutation $\overline{w}_{3,1} \in S_{28}$ is shown in Figure 3.

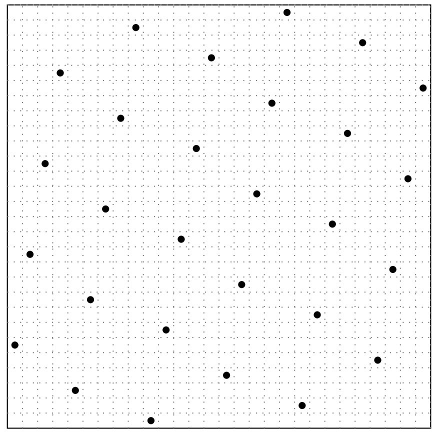

FIGURE 3. Plot of the permutation $\overline{w}_{3,1} \in S_{28}$ described in the proof of Theorem 3.4.

4. Conclusions and further directions

We have now shown that

$$\frac{k^2}{2} + 2k \leq \mathsf{MinProl}(k) \leq \frac{k^2}{2} + 3k + \frac{1}{2}$$

when k is odd, and

$$\frac{k^2}{2} + 2k \leq \mathsf{MinProl}(k) \leq \frac{k^2}{2} + 4k + 4$$

when k is even.

Example 4.1. Theorems 2.6 and 3.4 tell us that

$$16 \leq \mathsf{MinProl}(4) \leq 28.$$

In fact, $\mathsf{MinProl}(4) \leq 17$, as demonstrated by the permutation

$$5\ 10\ 15\ 2\ 7\ 12\ 17\ 4\ 9\ 14\ 1\ 6\ 11\ 16\ 3\ 8\ 13 \in S_{17}.$$

It would, of course, be ideal if Question 1.10 could be answered exactly, but our bounds do not leave a lot of room to maneuver, meaning that we are close to such an answer.

The concept of prolific permutations can be easily transferred to other contexts. For example, one could look at certain interesting subposets of $\mathcal{P}$ (for example, noteworthy pattern classes) and analyze k-prolific permutations in those settings. One could also adapt this question to other pattern-dependent structures and their hierarchies, thus providing another perspective to analyzing these families of objects.

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