Counting Patterns:
Equipopularity in Permutation Classes

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2016

Plotting Permutations

Definition

If π is a permutation of length n, then the *plot* of π is the set of points

$$\{(1, \pi(1)), (2, \pi(2)), \cdots (n, \pi(n))\} \subset \mathbb{R}^2$$

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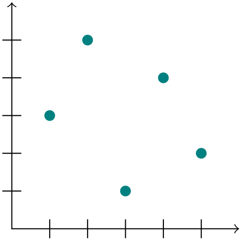

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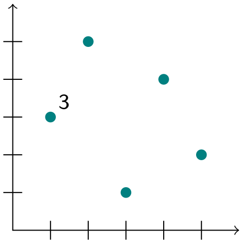

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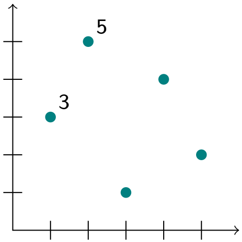

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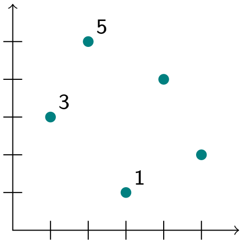

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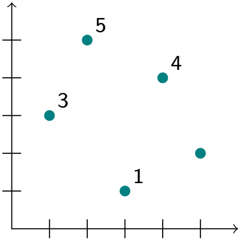

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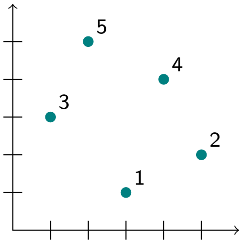

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Dots on a Plane

Definition

Let A and B be two sets of n points in $\mathbb{R}^2$, each with the property that no two points lie on the same horizontal or vertical line.
Say that A is *order isomorphic* to B (denoted $A \sim B$) if A can be transformed into B by stretching, contracting, and translating the axes horizontally and vertically.

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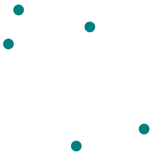
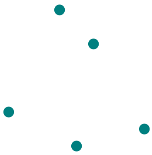


$\sim$



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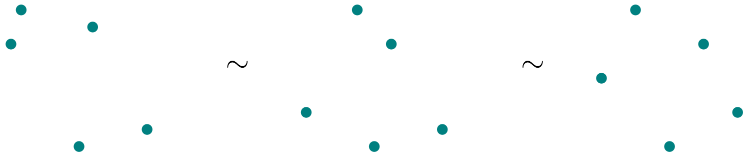

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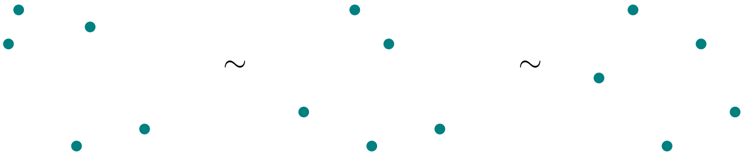

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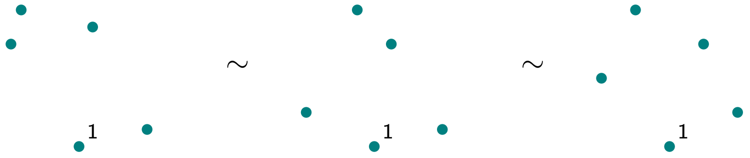

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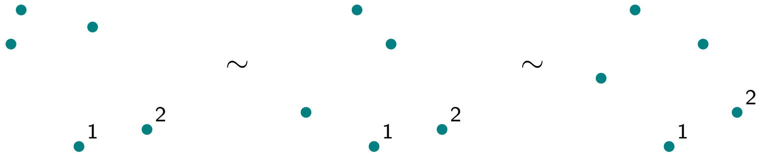

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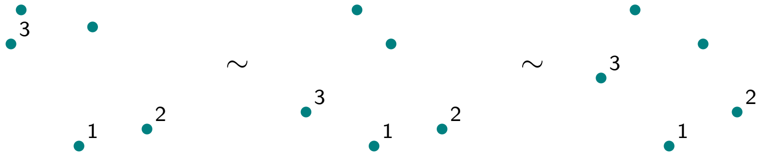

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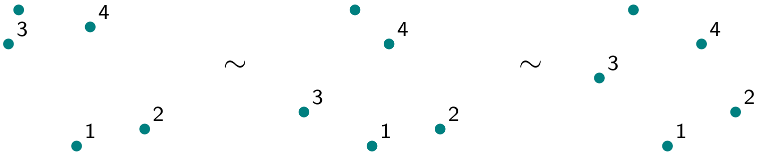

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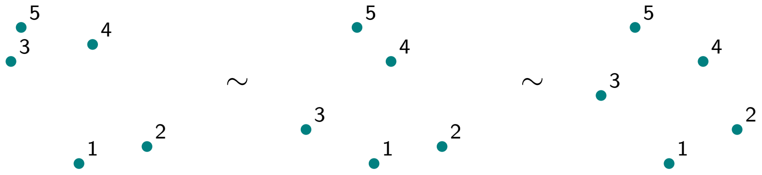

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Pattern Occurrences

Patterns

Say that one permutation π contains another permutation σ *as a pattern* (denoted $\sigma \prec \pi$) if the plot of π contains a subset which is equivalent to the plot of σ. The number of occurrences of σ in π (denoted $\nu_\sigma(\pi)$) is the number of such subsets.

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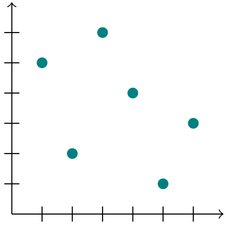

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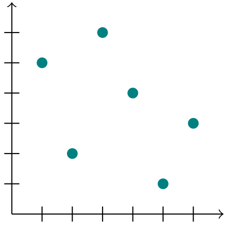

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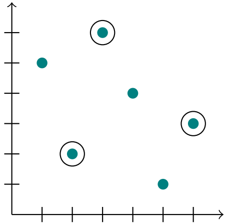

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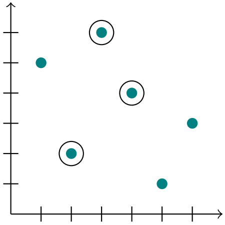

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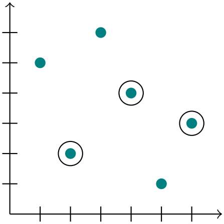

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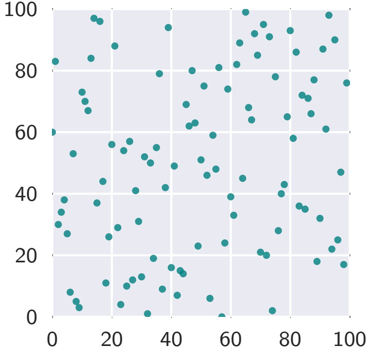

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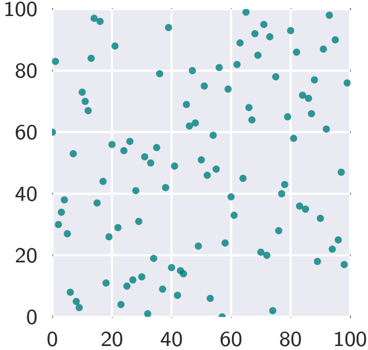

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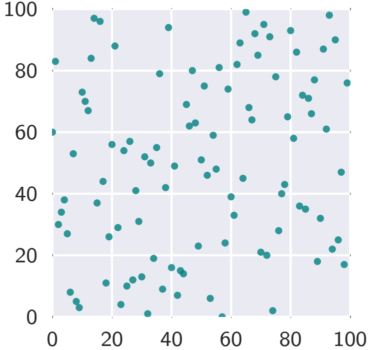

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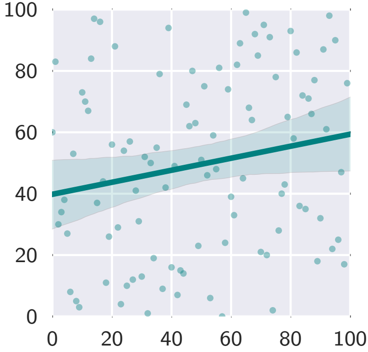

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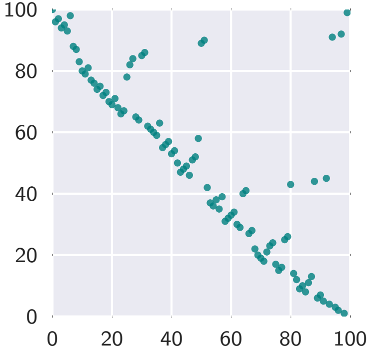

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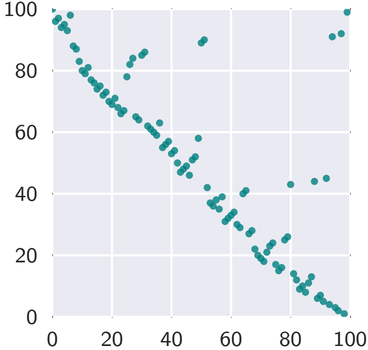

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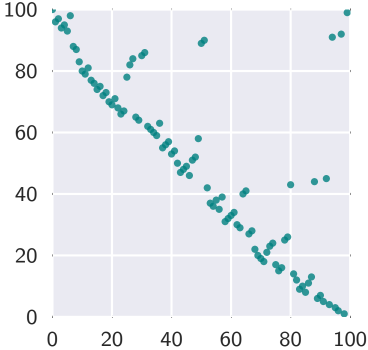

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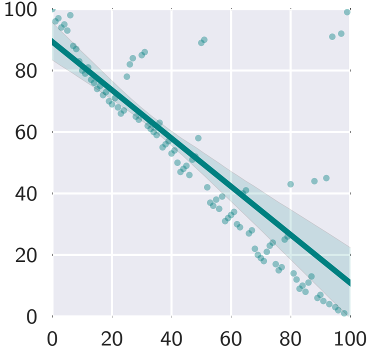

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Patterns as Random Variables

Theorem (Bóna 2007)

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Theorem (Janson, Nakamura, Zeilberger 2013)

For a randomly selected permutation of length n and two patterns σ and ρ, the random variables ν_σ and ν_ρ are asymptotically jointly normally distributed as $n \to \infty$.

Motivation

Fact

In $\mathfrak{S}_n$, the number of occurrences of a specific pattern depends only on the length of the pattern. That is, for a pattern $\sigma \in \mathfrak{S}_k$, we have

$$\nu_\sigma(\mathfrak{S}_n) = \frac{n!}{k!}\binom{n}{k}.$$

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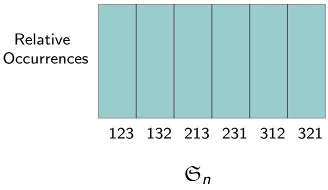

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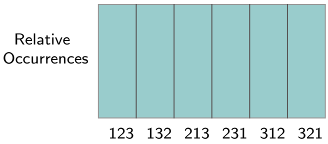
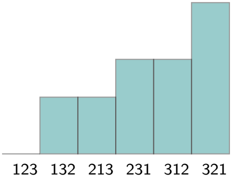


$\mathfrak{S}_n$



$\mathrm{Av}_n\,123$

Equipopularity

Definition

The *popularity* of a pattern σ in a class C is equal to

$$\sum_{n \geq 1} \nu_\sigma(C_n).$$

Definition

Patterns are said to be *equipopular* if they have the same number of occurrences (within a specified set or across two different sets).

Equipopularity — Example

Fact

For a class C and a pattern σ, we have

$$\nu_\sigma(C_n) = |\{(\pi;\sigma) : \pi \in C_n,\ \sigma \prec \pi\}|.$$

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Proposition

In the class $\mathrm{Av}(132)$, σ and σ^{-1} are equipopular.

Proof.

This follows from the fact that π avoids 132 if and only if π^{-1} avoids 132, and the fact that $\sigma \prec \pi$ if and only if $\sigma^{-1} \prec \pi^{-1}$. □

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Theorem (Chua, Sankar 2013)

If two patterns are equipopular in Av(132), then they *have the same structure*.

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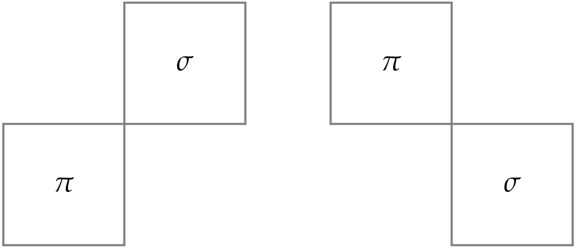
Theorem (Albert, H, Pantone)

Two patterns are equipopular in the separables if and only if they *have the same structure*. Further, the equipopularity classes are in bijection with the set of integer partitions.

Separable Permutations

Definition

Given two permutations π and σ, their *direct sum* ($\pi \oplus \sigma$) and *skew sum* ($\pi \ominus \sigma$) are defined as follows:



$\pi \oplus \sigma$ $\quad$ $\pi \ominus \sigma$

Alternate Definition

The separable permutations are those which can be constructed via arbitrary skew and direct sums of the permutation 1.

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Example

The permutation $\pi = 215643798$ is separable, since

$$\pi = \Big(1 \ominus 1\Big) \oplus \Big((1 \oplus 1) \ominus 1 \ominus 1\Big) \oplus 1 \oplus \Big(1 \ominus 1\Big).$$

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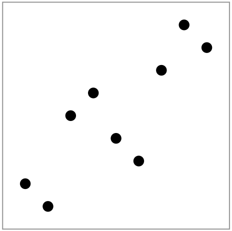

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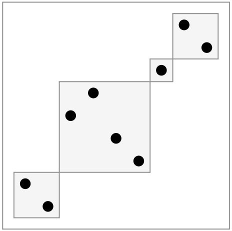

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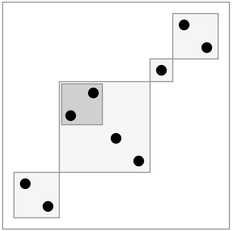

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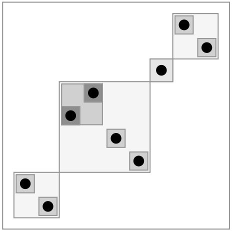

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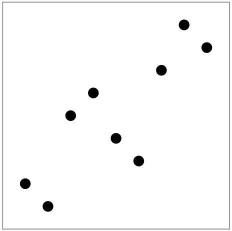

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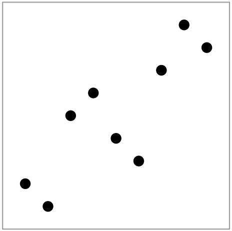

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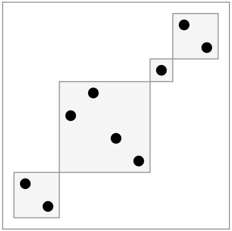

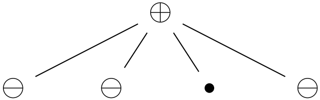

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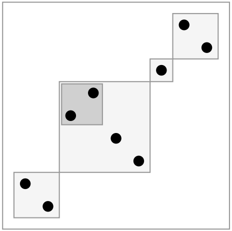

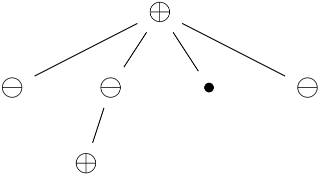

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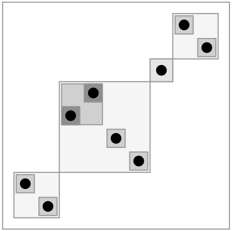

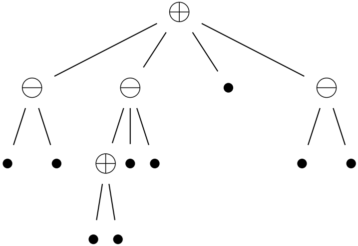

Tree Containment

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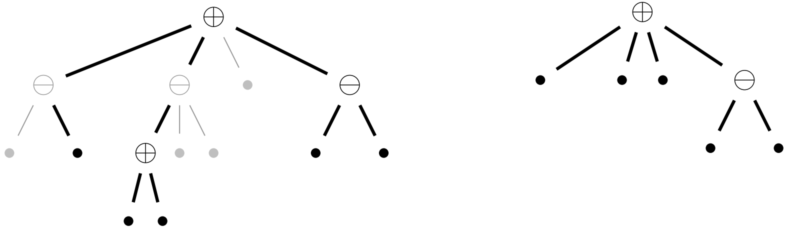

Equipopularity

Question

If two patterns are equipopular, how are their trees related?

Strategy

Strategy

Part 1

Find the operations on trees which preserve popularity.

Strategy

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Find the operations on trees which preserve popularity.

Part 2

Show that equipopularity implies that their trees are related by one of these operations.

Preserving Popularity

Preserving Popularity

Symmetries

Permutation	Tree

Preserving Popularity

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Complement	Flip signs

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Fact

If two permutations (or trees) are related by any of the above symmetries, then they are equipopular.

Preserving Popularity - Shuffling

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Lemma

Rearranging the children of any node preserves equipopularity.

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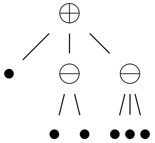

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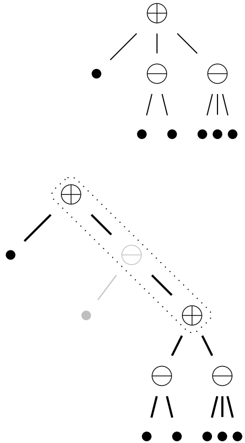

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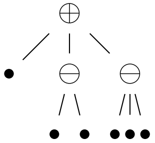

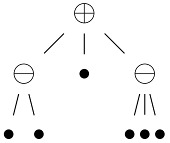

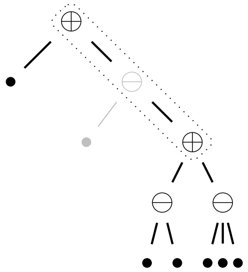

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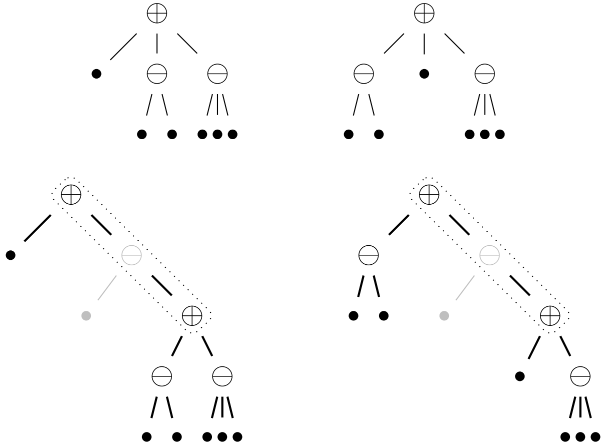

Preserving Popularity - Rotation

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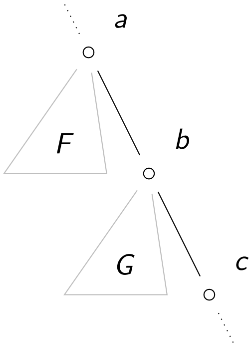

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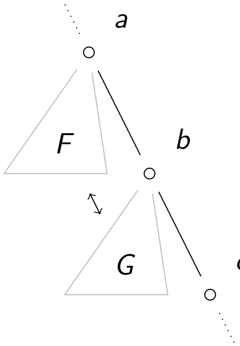

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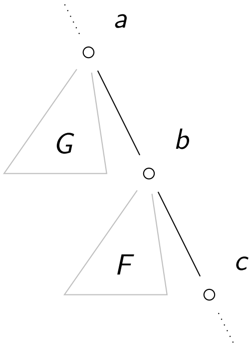


Preserving Popularity

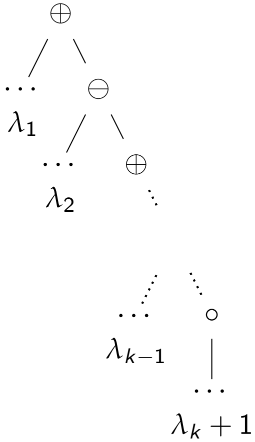
Lemma (Albert, H, Pantone)

The following operations preserve popularity:

- Reversal
- Complementation
- Inversion
- Shuffling
- Rotation

Canonical Representatives

Canonical Representatives



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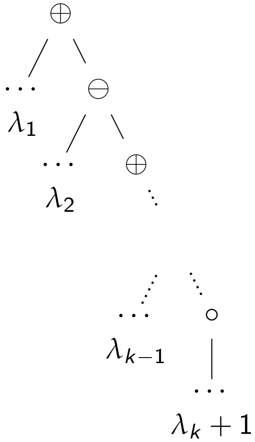

$$\lambda := \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

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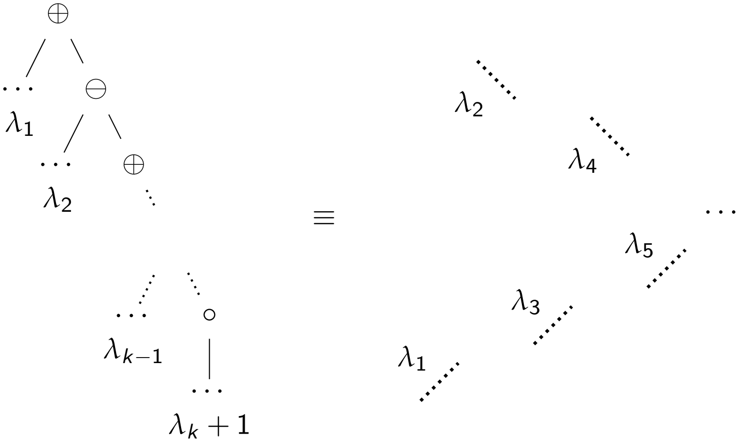

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If two patterns are equipopular, one can be transformed into the other by the above operations.

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If two patterns are equipopular, one can be transformed into the other by the above operations.

Corollary

The set of equipopularity classes for patterns of length n are in bijection with the set of partitions of the integer $n - 1$.

Rough Sketch of Proof

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Idea # 1

Given any arbitrary pattern, we can factor its popularity generating function into the popularity generating functions for monotone runs.

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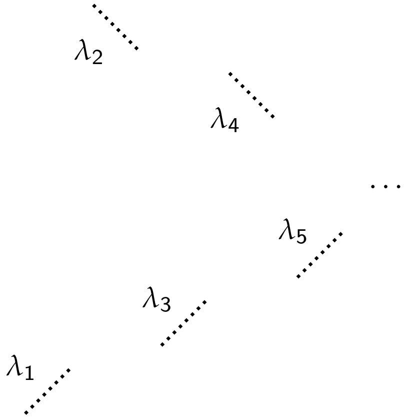

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- Notice (or let Sage/Maple/Mathematica/Singular tell you) that these are related to the *Gegenbauer polynomials*, a family of orthogonal polynomials.

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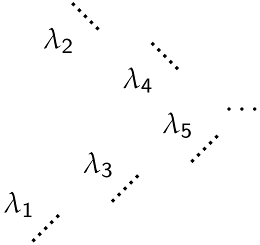
- Notice (or let Sage/Maple/Mathematica/Singular tell you) that these are related to the *Gegenbauer polynomials*, a family of orthogonal polynomials.
- Use the orthogonality of these polynomials to uniquely factor any product.

Conclusion



$$\leftrightarrow \lambda_1 + \lambda_2 + \cdots + \lambda_{k-1} + (\lambda_k - 1)$$

Length n canonical representative $\leftrightarrow$ partition of the integer $n - 1$.

Questions?