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Contents

Chapter 1

Vectors and Geometry of Planes and Curves

1.1 Vectors

Definition 1.1.1. A *vector* is a quantity that has both a magnitude and a direction. One exception is the zero vector $\mathbf{0}$ that has magnitude 0 but no direction.
A *scalar* is a quantity that has magnitude only.

Vectors will be denoted in boldface or (when using handwriting) with arrows above them. We represent 2-dimensional vectors as ordered pairs and 3-dimensional vectors as ordered triples as follows

$$\mathbf{u} = \langle u_1, u_2 \rangle \tag{1.1}$$

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle. \tag{1.2}$$

In equation (1.1) and (1.2), the real numbers (scalars) u_1, u_2 and v_1, v_2, v_3 are called the *components* of the vectors $\mathbf{u}$ and $\mathbf{v}$ respectively. A 2-dimensional vector has two components. A three-dimensional vector has three components.

Remark 1.1.2. Most of the illustrations and definitions I make will be for three-dimensional vectors, but the same concepts will also hold for two-dimensional vectors.

Geometrically, it is useful to think of vectors as directed line segments (arrows). The length of the arrow is the vector's magnitude, and the direction the arrow points is the vector's direction. This concept will be made more precise below.

Definition 1.1.3. The zero vector is the vector whose components are all (the scalar) zero. In two dimensions $\mathbf{0} = \langle 0, 0 \rangle$. In three dimensions, $\mathbf{0} = \langle 0, 0, 0 \rangle$. The zero-vector is the only vector that has no direction.

Remark 1.1.4. Note that $\mathbf{0} \neq 0$ since the object on the left-hand is a vector and the object on the right-hand side is a scalar. In this class, it will be crucial that you keep track of what kind of objects (e.g vectors or scalars) you are dealing with at all times.

Equality of Vectors

Two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are equal if and only if their corresponding components are equal. That is,

$$\mathbf{u} = \mathbf{v} \qquad \Leftrightarrow \qquad u_1 = v_1,\ u_2 = v_2,\ u_3 = v_3.$$

Notice in particular that if two vectors are equal, they must have the same number of components.

Example 1.1.5 (Constructing Vectors from pairs of points)**.** One way to construct vectors is from a pair of points. Let $P = (1, 5, 0)$ and $Q = (3, 9, 11)$ be points in $\mathbb{R}^3$. Find a $\vec{PQ}$, the vector that starts at P and ends at Q.

Solution

Remark 1.1.6. The starting point of a vector is immaterial. What matters is both the starting and the ending points (at the same time) of the vector. In example 1.1.5, The vector $\langle -2, 4, 11\rangle$ can just as well be thought of as the vector that starts at $(0, 0, 0)$ and ends at $(-2, 4, 11)$ or as the vector that starts at $(3, 0, -5)$ and ends at $(1, 4, 6)$. In this sense, we say that vectors are moveable.

Definition 1.1.7 (Magnitude of a Vector). Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$. The magnitude (also called the length or the norm) of $\mathbf{u}$, denoted $\|\mathbf{u}\|$, is given by

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

Example 1.1.8. Find the magnitude of $\vec{PQ}$ from example 1.1.5. How is the norm of a vector related to the three-dimensional Pythagorean Theorem?

Solution

Remark 1.1.9.

1. Often times it is easier to work with $\|\mathbf{u}\|^2$ than to work with $\|\mathbf{u}\|$ directly.
2. According to the definition of the norm of a vector, $\|\mathbf{0}\| = 0$.
3. The norm of a vector is a scalar.

Exercises

1. Let $P = (1, 1, -1)$ and $Q = (4, 7, 12)$.
   (a) Find $\overrightarrow{PQ}$, the vector that starts at P and ends at Q.
   (b) Find $\overrightarrow{QP}$, the vector that starts at Q and ends at P.
   (c) Make a guess as to how $\left\|\overrightarrow{PQ}\right\|$ and $\left\|\overrightarrow{QP}\right\|$ are related then compute both $\left\|\overrightarrow{PQ}\right\|$ and $\left\|\overrightarrow{QP}\right\|$.

1.1.1 Elementary Operations on Vectors

The basic precalculus-level operations on vectors are the following

- scaling
- adding and subtracting
- dot product
- cross product
- triple product
- Notice that division of vectors is not on the list

Scaling

Definition 1.1.10 (Scaling Vectors). If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is a vector and c is a scalar we can multiply $\mathbf{u}$ by c to get another vector $c\mathbf{u}$ as follows

$$c\mathbf{u} = c \langle u_1, u_2, u_3 \rangle = \langle cu_1, cu_2, cu_3 \rangle .$$

This operation is called *scalar multiplication* or *scaling.*

Remark 1.1.11. Keep the following in mind when scaling vectors

$$\text{(scalar)(vector)} = \text{vector}.$$

That is, a scalar multiple of a vector is a vector.

Example 1.1.12. Let $\mathbf{u} = \langle 3, -5, -1 \rangle$.

1. Compute $\|\mathbf{u}\|$
2. Compute both $2\mathbf{u}$ and $\|2\mathbf{u}\|$.
3. Compute both $-\frac{1}{2}\mathbf{u}$ and $\left\|-\frac{1}{2}\mathbf{u}\right\|$.

What effect does scaling a vector have on the length of a vector?

Solution

The next proposition says that the phenomenon we observed in example 1.1.12 holds in general.

Proposition 1.1.13. *If c is a scalar and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is a vector, then*

$$\|c\mathbf{u}\| = |c|\,\|\mathbf{u}\|. \tag{1.3}$$

Definition 1.1.14. Two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are called *parallel* if they are scalar multiples of each other. That is, $\mathbf{u}$ and $\mathbf{v}$ are parallel if there is a scalar $c \in \mathbb{R}$ such that

$$\mathbf{u} = c\mathbf{v}. \tag{1.4}$$

In terms of the components of $\mathbf{u}$ and $\mathbf{v}$ this says that $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if there is a scalar c such that the following equalities simultaneously hold

$$u_1 = cv_1 \qquad u_2 = cv_2 \qquad u_3 = cv_3.$$

Remark 1.1.15. Some people require the c is the definition of parallel vectors to be positive. This is purely a matter of convention. Keep in mind, however that in equation (1.4), if c is positive, then $\mathbf{u}$ and $\mathbf{v}$ point in the same direction and if c is negative then $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions.

Example 1.1.16. Determine whether the vectors $\langle -7, 5, 12 \rangle$ and $\langle 4, 4, 1 \rangle$ are parallel.

Solution

Example 1.1.17. Find a vector $\mathbf{v}$ parallel to $\mathbf{u} = \langle 6, -8, 0 \rangle$ whose length is 1.

Solution

Remark 1.1.18. Scaling a vector by a positive scalar can only affect the vector's magnitude, not its direction.

Remark 1.1.19. The vector found in example 1.1.17 is called a *unit vector* since its length is 1. If $c = 1/10$ (as opposed to $c = -1/10$) in this example, then we write $\mathbf{v} = \widehat{\mathbf{u}}$.

Definition 1.1.20 (Unit Vectors). A *unit vector* (also called a *direction vector*) is a vector whose length is one. That is, $\mathbf{u}$ is a unit vector if and only if $\|\mathbf{u}\| = 1$. As a notational convention, any vector written with "hat" over it has length one (i.e. is a unit vector).

Proposition 1.1.21. *Every nonzero vector* $\mathbf{u}$ *can be represented in the form*

$$\mathbf{u} = c\mathbf{v},$$

where c *is a positive scalar and* $\mathbf{v}$ *is a direction vector for* $\mathbf{u}$. *In fact, this may be achieved with* $c = \|\mathbf{u}\|$ *and* $\mathbf{v} = \mathbf{u}/\|\mathbf{u}\|$ *as follows*

$$\mathbf{u} = \|\mathbf{u}\| \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

Using (1.3) *with* $c = 1/\|\mathbf{u}\|$, *it easy to see that*

$$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = \left| \frac{1}{\|\mathbf{u}\|} \right| \|\mathbf{u}\| = \frac{\|\mathbf{u}\|}{\|\mathbf{u}\|} = 1.$$

Moreover, since $\mathbf{u}/\|\mathbf{u}\|$ *is a positive-scalar multiple of* $\mathbf{u}$, *we deduce* $\mathbf{u}/\|\mathbf{u}\|$ *has the same direction as* $\mathbf{u}$ *(hence the name "direction vector").*

Remark 1.1.22. Often you will need to perform the following task. Given a non-zero vector $\mathbf{u}$ (whose length may not be 1), find $\widehat{\mathbf{u}}$, a vector whose direction is the same (or possibly opposite) as $\mathbf{u}$'s direction, and whose length is a given positive number. (In example 1.1.17, the given positive number is 1). To do this follow these steps:

1. Compute $\|\mathbf{u}\|$.
2. Scale $\mathbf{u}$ by the reciprocal of $\|\mathbf{u}\|$ to get $\widehat{\mathbf{u}}$. That is

$$\widehat{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|}\mathbf{u}.$$

3. multiply $\widehat{\mathbf{u}}$ by the desired length.

The following two examples illustrate this process.

Example 1.1.23. Let $\mathbf{u} = \langle 1, 1, -1 \rangle$. Compute $\widehat{\mathbf{u}}$.

Solution

Example 1.1.24. Find a vector $\mathbf{v}$ whose length is 8 and whose direction is opposite the direction of $\mathbf{u} = \langle -2, 7, 1 \rangle$.

Solution

Addition and Subtraction

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. We add $\mathbf{u}$ and $\mathbf{v}$ *componentwise* as follows:

$$\mathbf{u} + \mathbf{v} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle .$$

Remark 1.1.25.

1. Geometrically, one can view vector addition as the "parallelogram rule" or the "tip-to-tail rule". Here is a picture:

2. Vector addition is useful when there are multiple forces acting on an object. To find the resultant force (i.e. the net force) acting on the object, simply represent all forces as vectors and add the vectors.

Example 1.1.26. Let $\mathbf{u} = \langle 5, 11, 12 \rangle$ and $\mathbf{v} = \langle -7, 1, -1 \rangle$. Compute $\|\mathbf{u} + \mathbf{v}\|$ and $\|\mathbf{u}\| + \|\mathbf{v}\|$. Compare your answers.

Solution

Definition 1.1.27 (Standard Basis Vectors). The vectors

$$\widehat{\mathbf{i}} = \langle 1, 0, 0 \rangle , \qquad \widehat{\mathbf{j}} = \langle 0, 1, 0 \rangle \qquad \text{and} \qquad \widehat{\mathbf{k}} = \langle 0, 0, 1 \rangle$$

are called the *standard basis* vectors (for $\mathbb{R}^3$). This is because every 3-dimensional vector can be decomposed uniquely as sums of scalar multiples of $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$ and $\widehat{\mathbf{k}}$. Observe that each of $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$ and $\widehat{\mathbf{k}}$ are unit vectors.

Example 1.1.28.

$$\langle 4, -2, 1\rangle = 4\widehat{\mathbf{i}} - 2\widehat{\mathbf{j}} + \widehat{\mathbf{k}}.$$

Exercises

1. Let $\mathbf{u} = \langle 4, -1, 6\rangle$. Compute $\widehat{\mathbf{u}}$.
2. Find a vector of length 5 whose direction is opposite the direction of $\langle -1, 4, -2\rangle$.
3. Find real numbers a and b such that the vectors $\mathbf{u} = \langle 6, -4, 3\rangle$ and $\mathbf{v} = \langle a, -1, 2b\rangle$ are parallel. Once a and b have been found, express $\mathbf{v}$ as a scalar multiple of $\mathbf{u}$.
4. Let $\mathbf{u} = \langle 1, 4, -3\rangle$ and $\mathbf{v} = \langle 6, 8, 1\rangle$. Find a vector $\mathbf{w}$ that is parallel to $2\mathbf{u} - \mathbf{v}$ and whose length is 9. Compute the length of your answer.

Dot Product

Definition 1.1.29 (Dot Product). Given vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, we define the dot product of $\mathbf{u}$ and $\mathbf{v}$ by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Remark 1.1.30. Observe that the dot product of two vectors is a scalar. That is, the dot product obeys the following

$$(\text{vector}) \cdot (\text{vector}) = \text{scalar}.$$

Example 1.1.31. Let $\mathbf{u} = \langle 1, -6, 7 \rangle$, $\mathbf{v} = \langle -2, 5, 2 \rangle$ and $\mathbf{w} = \langle 4, 2, -1 \rangle$. Compute $\|(\mathbf{u} \cdot \mathbf{v})\mathbf{w}\|$.

Solution

Proposition 1.1.32 (Algebraic Properties of Dot Product)**.** *Let* $\mathbf{u}$*,* $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors and let* c *be a scalar. The following properties hold.*

1. *commutative*
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$
2. *distributes over vector addition*
$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$
3. *Associativity:*
$$(c\mathbf{u}) \cdot \mathbf{v} = c\ (\mathbf{u} \cdot \mathbf{v}) \qquad \textit{and} \qquad \mathbf{u} \cdot (c\mathbf{v}) = c\ (\mathbf{u} \cdot \mathbf{v}).$$
4. *Cauchy-Schwarz inequality:*
$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\, \|\mathbf{v}\|\,.$$
Notice that on the left-hand side of this inequality, the usual absolute values are used (not the norm). This is because $\mathbf{u} \cdot \mathbf{v}$ *is a scalar.*

Using the dot product, we can also derive the following inequality which was eluded to in example 1.1.26.

Proposition 1.1.33 (Triangle Inequality). *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors. Then*

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Angle Between Vectors

One of the most useful features of the dot product is that given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, the dot product "knows" the angle between $\mathbf{u}$ and $\mathbf{v}$. In particular, if θ is (an appropriately chosen) angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta. \tag{1.5}$$

The following special cases of equation (1.5) are of particular importance.

1. If $\theta = \pi/2$ (so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal) then
$$\mathbf{u} \cdot \mathbf{v} = 0.$$
2. If $\theta = 0$ (so $\mathbf{u}$ and $\mathbf{v}$ point in the same direction) then
$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\|.$$
3. If $\theta = \pi$ (so $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions) then
$$\mathbf{u} \cdot \mathbf{v} = -\|\mathbf{u}\| \, \|\mathbf{v}\|.$$

Example 1.1.34. Let $\mathbf{u} = \langle 3, -1, 6 \rangle$ and $\mathbf{v} = \langle 2, 5, -3 \rangle$.

1. Find the non-obtuse angle between $\mathbf{u}$ and $\mathbf{v}$.
2. Find the angle between $\mathbf{u}$ and $\widehat{\mathbf{u}}$.
3. Find a vector of length 4 that is orthogonal to $\mathbf{v}$ (there are many).

Solution

Example 1.1.35. Find the cosine of the angle between the vectors $\langle 0, 4, 3\rangle$ and $\langle -3, 0, 4\rangle$.

Solution

Work

Definition 1.1.36. The work W done by a force $\mathbf{F}$ in moving an object a (directed) distance $\mathbf{d}$ is

$$W = \mathbf{F} \cdot \mathbf{d}.$$

Remark 1.1.37. Work has no direction, it is a scalar.

Example 1.1.38. A wagon is being pulled with a force $\mathbf{F}$ of magnitude 100N at an angle of $\pi/4$ to the horizontal. Find the work done by $\mathbf{F}$ in pulling the wagon 10m.

Solution

Example 1.1.39. Let $\mathbf{F} = \langle 1, 1, 2 \rangle$ be a constant force (whose components are measured in Newtons). Find the work done by $\mathbf{F}$ in moving an object 3m along the line $y = \sqrt{3}x$ in the xy-plane.

Solution

Decomposition of Vectors

Consider two vectors $\mathbf{u}$ and $\mathbf{v}$. One of the basic skills you should have is to be able to decompose $\mathbf{u}$ into the sum of two vectors, one that is parallel (or possibly anti-parallel) to $\mathbf{v}$ and one that is orthogonal to $\mathbf{v}$. That is, you should write $\mathbf{u}$ as

$$\mathbf{u} = \mathbf{a} + \mathbf{b} \tag{1.6}$$

such that both

$$\mathbf{a} \cdot \mathbf{v} = 0 \qquad \text{and} \qquad \mathbf{b} = c\mathbf{v} \tag{1.7}$$

where c is an appropriate scalar. The tool for accomplishing this is the dot product.

Definition 1.1.40 (Projection and Scalar Projection)**.** Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors. The (vector) orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} \right) \frac{\mathbf{v}}{\|\mathbf{v}\|}. \tag{1.8}$$

The scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is called the scalar projection of $\mathbf{u}$ onto $\mathbf{v}$ and is given by

$$\operatorname{scal}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} \tag{1.9}$$

Remark 1.1.41.

1. $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is a vector, while $\operatorname{scal}_{\mathbf{v}} \mathbf{u}$ is a scalar.
2. Observe that $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is parallel to $\mathbf{v}$ despite it being called the "orthogonal" projection of $\mathbf{u}$ onto $\mathbf{v}$.
3. Notice that in view of equations (1.8) and (1.9), we have both

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = (\operatorname{scal}_{\mathbf{v}} \mathbf{u})\, \widehat{\mathbf{v}}. \qquad \text{and} \qquad \operatorname{scal}_{\mathbf{v}} \mathbf{u} = \|\operatorname{proj}_{\mathbf{v}} \mathbf{u}\|.$$

Define

$$\operatorname{orth}_{\mathbf{v}} \mathbf{u} = \mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}.$$

Lemma 1.1.42. $\mathrm{orth}_{\mathbf{v}}\,\mathbf{u}$ *is orthogonal to* $\mathbf{v}$.

Proof

We have succeeded in finding a decomposition of $\mathbf{u}$ of the form given in equations (1.6) and (1.7). That is

$$\mathbf{u} = \mathrm{proj}_{\mathbf{v}}\,\mathbf{u} + \mathrm{orth}_{\mathbf{v}}\,\mathbf{u}$$

with $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$ parallel to $\mathbf{v}$ and $\mathrm{orth}_{\mathbf{v}}\,\mathbf{u}$ orthogonal to $\mathbf{v}$.

Example 1.1.43. Let $\mathbf{u} = \langle 5, 1, -5\rangle$ and $\mathbf{v} = \langle -1, 1, -2\rangle$. Compute the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$.

Solution

1.1.2 Exercises

1. A force $\mathbf{F} = \langle 3, 2, 7\rangle$, whose components are measured in Newtons, moves an object from the point $P = (5, 5, 0)$ to the point $Q = (1, -2, 5)$, where distance is measured in meters. Find the work done by the force.

2. A sled is pushed 5m along a sidewalk by a force of magnitude 10N applied at an angle of $\pi/6$ to the horizontal. Find the work done by the Force.

3. Find the non obtuse angle between the vectors $\langle 3, 0, \sqrt{3}\rangle$ and $\langle 1, 0, \sqrt{3}\rangle$.

4. Let $\mathbf{u} = \langle 5, 1, -5\rangle$ and $\mathbf{v} = \langle -1, 1, 2\rangle$.

   (a) Compute $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$, $\operatorname{scal}_{\mathbf{v}} \mathbf{u}$ and $\operatorname{orth}_{\mathbf{v}} \mathbf{u}$.

   (b) Compute both
   $$\operatorname{proj}_{\mathbf{v}}(\operatorname{proj}_{\mathbf{v}} \mathbf{u}) \quad \text{and} \quad \operatorname{proj}_{\mathbf{v}}(\operatorname{orth}_{\mathbf{v}} \mathbf{u}).$$
   Explain the geometrical significance of your answers.

   (c) Find a unit vector $\widehat{\mathbf{w}}$ that is orthogonal to $\mathbf{v}$ and such that $\mathbf{u} \cdot \mathbf{v} > 0$.

5. Find all real numbers a such that the vectors $\langle 1, a, 2a\rangle$ and $\langle 8, -16, 4\rangle$ are orthogonal.

6. A constant force $\mathbf{F} = \langle 4, 3, 2\rangle$ in Newtons moves an object from $(1, 4, 7)$ to $(2, 4, 1)$. Find the work done by the force.

7. Let $\mathbf{u} = \langle -1, 2, 3\rangle$ and $\mathbf{v} = \langle -1, 1, 4\rangle$. Find vectors $\mathbf{p}$ and $\mathbf{n}$ such that $\mathbf{p}$ is parallel to $\mathbf{v}$, $\mathbf{n}$ is orthogonal to $\mathbf{v}$ and $\mathbf{u} = \mathbf{p} + \mathbf{n}$.

8. Prove the following statement. If A, B and C are points on a circle of radius 1 such that segment $\overline{AB}$ is a diameter, then triangle $\triangle ABC$ is a right triangle.

Cross Product

Recall the determinant of a 2×2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Definition 1.1.44 (Cross Product). Given two (three-dimensional) vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, we define the cross product $\mathbf{u} \times \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ by

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \widehat{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \widehat{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \widehat{\mathbf{k}} \\ &= \langle u_2v_3 - u_3v_2,\ -(u_1v_3 - u_3v_1),\ u_1v_2 - u_2v_1 \rangle . \end{aligned}$$

Remark 1.1.45. The cross product of two vectors is a vector:

$$(\text{vector}) \times (\text{vector}) = \text{vector}.$$

Remark 1.1.46. The crosss product is only defined for three-dimensional vectors. To compute the cross product of two-dimensional vectors, simple write the two-dimensional vectors as three-dimensional vectors with third component zero then compute the cross product as usual.

Remark 1.1.47. The formula for the cross product can be easily memorized as the formal determinant

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example 1.1.48. Compute the cross product $\langle 3, 2, -1\rangle \times \langle 2, -5, 0\rangle$.

Solution

Here are the algebraic properties of the cross product

Proposition 1.1.49. *Let* $\mathbf{u}$*,* $\mathbf{v}$ *and* $\mathbf{w}$ *be three-dimensional vectors and let* c *be a scalar. The following algebraic properties hold.*

1. *Anticommutativity*
$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$$
2. *Associativity*
$$(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (c\mathbf{v}).$$
3. *Distributing over vector addition from the left*
$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}.$$
4. *Distributing over vector addition from the right*
$$(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}.$$

Geometry of the cross product

The cross product is a vector, so it has both a magnitude and direction (provided it is nonzero). The magnitude of the cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ is

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta,$$

where θ is the (smaller) angle between $\mathbf{u}$ and $\mathbf{v}$. Elementary computations show that the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ is also equal to $\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta$. That is

$$\|\mathbf{u}\times\mathbf{v}\| = \text{ area of parallelogram formed by } \mathbf{u} \text{ and } \mathbf{v}.$$

In particular, if $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\mathbf{u}\times\mathbf{v}=\mathbf{0}$. Here is a picture:

Now to determine the direction of $\mathbf{u} \times \mathbf{v}$. Consider the example

Example 1.1.50. Let $\mathbf{u} = \langle 6, 5, 1\rangle$ and $\mathbf{v} = \langle 5, -7, 1\rangle$.

1. Determine whether $\mathbf{u}$ and $\mathbf{v}$ are parallel.
2. Compute both
$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) \qquad \text{and} \qquad \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}).$$

Solution

Based on the results of example 1.1.50, what do you conclude about the direction of $\mathbf{u} \times \mathbf{v}$ in relation to the direction(s) of $\mathbf{u}$ and $\mathbf{v}$? How many vectors have this property and have length $\|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$?

The direction of $\mathbf{u} \times \mathbf{v}$ is determined by the right-hand rule.

Summary: Given vectors $\mathbf{u}$ and $\mathbf{v}$, the cross product $\mathbf{u} \times \mathbf{v}$ is a vector which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Its length is the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ and (if $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$). Its direction is determined by the right-hand rule.

Example 1.1.51. Let $\mathbf{u} = \langle 4, 3, -1 \rangle$ and $\mathbf{v} = \langle 6, 2, -1 \rangle$.

1. Determine whether $\mathbf{u}$ and $\mathbf{v}$ are collinear.
2. Find a vector $\mathbf{w}$ which is orthogonal to both $\mathbf{u}$ and $\mathbf{w}$ and whose length is the area of the triangle determined by $\mathbf{u}$ and $\mathbf{v}$.

Solution

Example 1.1.52. Find all unit vectors that are normal to both $\langle 3,2,0\rangle$ and $\langle 0,1,-1\rangle$.

Solution

A common application of the cross product is torque.

Definition 1.1.53. Let $\mathbf{F}$ be a force applied to the head of a vector $\mathbf{r}$ (with the base of $\mathbf{r}$ held at a fixed position). The torque τ is the (vector) twisting effect resulting from the application of the force and is defined by

$$\tau = \mathbf{r} \times \mathbf{F}.$$

Remark 1.1.54. If you think of $\mathbf{r}$, the vector whose head is having a force applied to it as the lever of a wrench , and the force as the force you apply to tighten a bolt, then the direction of the torque follows the "righty-tighty, lefty-loosey" rule.

Example 1.1.55. Suppose a wrench of length 1/4m is attached to a bolt and a force of magnitude 10N is applied to the end of the wrench opposite the bolt in a direction perpendicular to the wrench. Find the magnitude of the resulting torque.

Solution

1.1.3 Exercises

1. Find a vector that is orthogonal to both $\mathbf{u} = \langle 1, -5, 7\rangle$ and $\mathbf{v} = \langle 6, -6, 1\rangle$ and whose length is equal to the area of the triangle determined by $\mathbf{u}$ and $\mathbf{v}$.

2. Determine whether the vectors $\mathbf{u} = \langle 5, -4, 12\rangle$ and $\mathbf{v} = \langle 6, 11, -1\rangle$ are collinear. If the vectors are not collinear, find the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$. [Hint: You should be able to do this with a single computation.]

3. Find all vectors of length 5 that are orthogonal to both $\langle 3, -1, 9\rangle$ and $\langle 7, 8, 5\rangle$.

4. Let $\mathbf{u} = \langle 1, 1, 1\rangle$ and $\mathbf{v} = \langle 5, 1, 1\rangle$.

   (a) Compute $\mathbf{u} \times \mathbf{v}$.

   (b) Compute both $\text{proj}_{\mathbf{u}}(\mathbf{u} \times \mathbf{v})$ and $\text{proj}_{\mathbf{v}}(\mathbf{u} \times \mathbf{v})$. Explain why your answer makes sense using geometrical reasoning.

   (c) Compute both $\text{orth}_{\mathbf{u}}(\mathbf{u} \times \mathbf{v})$ and $\text{orth}_{\mathbf{v}}(\mathbf{u} \times \mathbf{v})$.

   (d) Now let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors (not necessarily the given vectors). Based on your answers to the previous parts, what are the values of $\text{proj}_{\mathbf{u}}(\mathbf{u} \times \mathbf{v})$, $\text{proj}_{\mathbf{v}}(\mathbf{u} \times \mathbf{v})$, $\text{orth}_{\mathbf{u}}(\mathbf{u} \times \mathbf{v})$ and $\text{orth}_{\mathbf{v}}(\mathbf{u} \times \mathbf{v})$?

5. Let $P = (1, 1, 5)$ and $Q = (6, 1, -1)$ and $R = (5, -3, 1)$. Use the cross product to compute the area of triangle PQR.

6. This problem has two parts.

   (a) Find an example of two unit vectors $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$ such that $\widehat{\mathbf{u}} \times \widehat{\mathbf{v}}$ is not a unit vector.

   (b) Find an example of two unit vectors $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$ such that $\widehat{\mathbf{u}} \times \widehat{\mathbf{v}}$ is a unit vector.

   (c) Suppose $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$ are unit vectors. Find a relation between $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$ that guarantees that $\widehat{\mathbf{u}} \times \widehat{\mathbf{v}}$ is a unit vector.

Triple Product

Example 1.1.56. Find the volume of the parallelepiped determined by three non-coplanar vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Hint: the volume V of a parallelepiped is

$$V = (\text{ area of base })(\text{ height }).$$

Solution

Definition 1.1.57 (Triple Product)**.** Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be (three dimensional) vectors. A triple product of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \tag{1.10}$$

Notice that we did not say *the* triple product. Other possible triple products come from permuting $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in the formula above.

Remark 1.1.58. In equation (1.10), there is no need for the parenthesis. The reason is because $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$ makes no sense; crossing a scalar (namely $\mathbf{u} \cdot \mathbf{v}$) with a vector is impossible.

Remark 1.1.59. The triple product is a scalar.

$$(\text{vector}) \cdot (\text{vector}) \times (\text{vector}) = \text{scalar}.$$

Proposition 1.1.60. *Let* $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$.

1. *The triple product may be computed via the determinant*

$$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_2 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

2. $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *are coplanar if and only if* $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = 0$.

The triple product is useful when finding distances between objects. In fact, the triple product allows us to find the distances between objects without using any optimization (derivative) theory. Using dot products, cross products or triple products as necessary, we will soon be able to compute distances between the following objects.

1. Distance between parallel lines
2. Distance between skew lines
3. Distance between parallel planes
4. Distance between a point an a plane
5. Distance between a line and a parallel plane

Before we can do any of this, we need to learn about lines and planes. First however, you should do some exercises to be sure you understand the triple product.

1.1.4 Exercises

1. Let $\mathbf{u} = \langle 4, -5, -3\rangle$, $\mathbf{v} = \langle 5, 6, 2\rangle$ and $\mathbf{w} = \langle 0, -5, -1\rangle$.
   (a) Compute $\mathbf{w} \cdot \mathbf{u} \times \mathbf{v}$.
   (b) With no further computation, compute both $\mathbf{w} \cdot \mathbf{v} \times \mathbf{u}$ and $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}$
2. Determine whether the vectors $\mathbf{u} = \langle 5, -4, 6\rangle$, $\mathbf{v} = \langle 5, 0, -5\rangle$ and $\mathbf{w} = \langle 7, 7, -2\rangle$ are coplanar. If the vectors are not coplanar, compute the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. [Hint: you should be able to do this with a single computation.]
3. Find all real numbers a such that the vectors $\langle 1, 2, 3\rangle$, $\langle -4, 2, 1\rangle$ and $\langle 3, -1, a\rangle$ are coplanar.
4. (a) Let $\mathbf{u} = \langle 1, 1, -3\rangle$, $\mathbf{v} = \langle 0, 1, 4\rangle$ and $\mathbf{w} = \langle -6, 0, 2\rangle$. Find all vectors $\mathbf{a}$ which is parallel to $\mathbf{w}$ and such that the volume of the parallelepiped whose adjacent sides are $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{a}$ is equal to the area of the parallelogram whose adjacent sides are $\mathbf{u}$ and $\mathbf{v}$ (ignoring units of course).
   (b) What is $|\mathrm{scal}_{\mathbf{u}\times\mathbf{v}}\, \mathbf{a}|$?
   (c) Explain the geometrical significance of the answer to the previous part.

1.2 Geometry of Lines and Planes

1.2.1 Lines

Consider the calc 1-version of a line

Example 1.2.1. Consider the line

$$y = 2x + 1. \tag{1.11}$$

This line passes through the point $(0, 1)$ and has slope 2. To write this in terms of vectors, we use the point $(0, 1)$ and the direction vector $\mathbf{v} = \langle 1, 2\rangle$, which is obtained from the slope. We can therefore represent the line in equation (1.11) in vector form as

$$\mathbf{r}(t) = \langle 0, 1\rangle + t\,\langle 1, 2\rangle \qquad -\infty < t < \infty.$$

This representation is called a parameterization of the line (with parameter t).

Definition 1.2.2 (Line). A *line* $\mathcal{L}$ with direction vector $\mathbf{v} \neq \mathbf{0}$ containing the point $P = (p_1, p_2, p_3)$ is the set of all points of the form

$$\begin{aligned} \mathbf{r}(t) &= \langle p_1, p_2, p_3 \rangle + t \langle v_1, v_2, v_3 \rangle, \qquad -\infty < t < \infty \\ &= \langle p_1 + tv_1, p_2 + tv_2, p_3 + tv_3 \rangle . \end{aligned} \tag{1.12}$$

Equation (1.12) is called the *vector equation* of the line.
By looking at the components of the line individually, we obtain the *parametric equations* of the line

$$\left\{ \begin{array}{l} x(t) = p_1 + tv_1 \\ y(t) = p_2 + tv_2 \\ z(t) = p_3 + tv_3 \end{array} \right\} \qquad -\infty < t < \infty.$$

By solving each of the parametric equations for t and equating the results, we get the *symmetric equations* of the line

$$\frac{x - p_1}{v_1} = \frac{y - p_2}{v_2} = \frac{z - p_3}{v_3},$$

provided v_1, v_2 and v_3 are nonzero.

Relative Orientations of Lines in Space

Given two lines embedded in three-dimensional space, there are three possibilities for their relative orientations. The lines can be parallel, the lines can intersect, or the lines can be skew. Of course, given two lines, you should be able to determine their relative orientations, and depending on their orientations, there will be further questions you should be able to answer. Here is a list of how to determine the relative orientations of two lines in space and in each case of relative orientation, how to answer the questions that follow. Examples are included to illustrate the concepts. For the following list, let us fix some notation. Let

$$\mathbf{r}(t) = \mathbf{u_0} + t\mathbf{u} \qquad -\infty < t < \infty \tag{1.13}$$

and

$$\mathbf{R}(s) = \mathbf{v_0} + s\mathbf{v} \qquad -\infty < s < \infty \tag{1.14}$$

be parameterizations for two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively.

Definition 1.2.3. We say $\mathbf{r}(t)$ and $\mathbf{R}(s)$ are *parallel* if their direction vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel.

Question of interest: What is the distance between parallel lines? This is easily determined by using the cross product along with the formula

$$\text{(Area of parallelogram)} = (\text{ length of base})(\text{Height}).$$

Example 1.2.4. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lines with parameterizations

$$\mathbf{r}(t) = \langle 1, 0, 0\rangle + t\,\langle -2, 1, 5\rangle \qquad -\infty < t < \infty$$

and

$$\mathbf{R}(s) = \langle 7, -9, 0\rangle + s\left\langle \frac{2}{5}, -\frac{1}{5}, -1\right\rangle \qquad -\infty < s < \infty.$$

respectively.

1. Show that $\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel.
2. Compute the distance between $\mathcal{L}_1$ and $\mathcal{L}_2$.

Solution

Definition 1.2.5. We say $\mathbf{r}(t)$ and $\mathbf{R}(s)$ are *intersecting* if they have a point in common. In terms of the parameterizations given in equations (1.13) and (1.14), this says there are "magic" parameter values t^* and s^* such that

$$\mathbf{u_0} + t^*\mathbf{u} = \mathbf{v_0} + s^*\mathbf{v}.$$

(This means the components of these vectors must all coincide when both $t = t^*$ and $s = s^*$).

Questions of interest for intersecting lines:

1. Find the angle between intersecting lines.
   The angle between two intersecting lines is simply the angle between the direction vectors of the lines. Luckily, we know how to find the angle between two vectors.

2. Find the point of intersection.

Example 1.2.6. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be given by

$$\begin{cases} x(t) &= 1 \\ y(t) &= t \\ z(t) &= -4 - 2t \end{cases} \qquad \text{and} \qquad x - 2 = -\frac{y+1}{3} = \frac{z-5}{13}$$

respectively, where $-\infty < t < \infty$. Given that $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect, find their point of intersection.

Solution

Definition 1.2.7. We say $\mathbf{r}(t)$ and $\mathbf{R}(s)$ are *skew* if they are neither parallel nor intersecting.

Question of interest for skew lines: What is the distance between skew lines? To do this, simply form an appropriate parallelepiped, find its volume and divide by the area of its base (you must choose the correct base).

Example 1.2.8. Consider the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ given by

$$x = y = z$$

and

$$x + 1 = \frac{y}{2} = \frac{z}{3}.$$

respectively. Show that $\mathcal{L}_1$ and $\mathcal{L}_2$ are skew and find the distance between them.

Solution

1.2.2 Exercises

1. Find the symmetric equations of the line that passes through the point $(-2, 3, 1)$ and is parallel to the line

$$\begin{cases} x(t) &= 1 \\ y(t) &= t - 3 \\ z(t) &= 1 - 3t. \end{cases}$$

2. Find the vector equation of the line that is orthogonal to both $\langle 6, -5, 1\rangle$ and $\langle 1, 0, 1\rangle$ and passes through the point $(0, 0, 0)$.

3. Show that the lines

$$\mathbf{r}_1(t) = \langle t, 1 - 5t, 4t\rangle \qquad \text{and} \qquad \mathbf{r}_2(t) = \left\langle 5 + \frac{t}{5}, 5 - t, 1 + \frac{4t}{5} \right\rangle$$

are parallel and find the distance between them.

4. Show that the lines

$$\mathbf{r}_1(t) = \langle 5, -5, 1\rangle + t\,\langle 5, -6, 1\rangle \qquad \text{and} \qquad \mathbf{r}_2(t) = \langle 0, 1, 0\rangle + t\,\langle 1, -5, 4\rangle$$

are intersecting. Find both the point of intersection and the angle between the lines.

5. Show that the lines

$$\mathbf{r}_1(t) = \langle 4, 0, 1\rangle + t\,\langle 1, 1, 0\rangle \qquad \text{and} \qquad \mathbf{r}_2(t) = \langle 0, 1, -5\rangle + t\,\langle 1, 2, -1\rangle$$

are skew and find the distance between them.

6. Find the symmetric equations of the line that passes through the point $(0, 7, 1)$ and is orthogonal to both of the lines $\mathbf{r}_1(t) = \langle 2 - t, 1 + t, 5\rangle$ and $\mathbf{r}_2(t) = \langle t, 1 - 3t, 4t - 5\rangle$.

7. Consider the lines

$$x = 1 + 2t,\ y = 3t,\ z = 2 - t \qquad -\infty < t < \infty$$

and

$$x + 1 = y - 4 = \frac{z - 1}{3}.$$

Determine whether the lines are parallel, intersecting or skew. If the lines are intersecting, find both the point of intersection and the angle between the lines. Otherwise, find the distance between the lines.

1.2.3 Planes

To determine a plane we need a point on the plane and a vector normal to the plane. Let $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ be a particular point on a plane $\mathcal{P}$ and let $\mathbf{n} = \langle n_1, n_2, n_3 \rangle$ be a vector normal to $\mathcal{P}$. Then a point $\mathbf{r} = \langle x, y, z \rangle$ is on $\mathcal{P}$ if and only if

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0. \tag{1.15}$$

Equation (1.15) is the equation of a plane. In terms of the components, this can be written either

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

or

$$n_1 x + n_2 y + n_3 z = d,$$

where

$$d = \mathbf{n} \cdot \mathbf{r}_0 = n_1 x_0 + n_2 y_0 + n_3 z_0.$$

Example 1.2.9. Find an equation of the plane containing the point $(3, -1, 1)$ and orthogonal to the vector $\langle 1, -1, 1 \rangle$.

Solution

Example 1.2.10. The following points are non collinear

$$P_1 = (-2, 3, 0) \quad P_2 = (1, 4, 3) \quad P_3 = (5, 2, -3).$$

Find an equation of the plane containing P_1, P_2 and P_3.

Solution

Relative positions of planes in 3-dimensional space
Given two planes in three-dimensional space, there are only two possibilities for their relative orientations. Either the planes are parallel or the planes intersect.

Definition 1.2.11. Two planes are *parallel* if their normal vectors are parallel.

If two planes are parallel, we should be able to find the distance between them.

Example 1.2.12. Show that the planes whose equations are

$$2x - 3y + z = 6 \qquad \text{and} \qquad -6x + 9y - 3z = 2$$

are parallel. Find the distance between the planes.

Solution

If two planes in three-dimensional space are non parallel, they must intersect. If the planes are not the same plane, the intersection will form a line. Questions of interest in this case:

1. Find a parametric equation for the line determined by the intersection of two planes.

2. Find the angle between intersecting planes.

3. Find an equation of the plane orthogonal to both of the given planes.

Example 1.2.13. Consider the planes whose equations are

$$2x - 3y + z = 6 \qquad \text{and} \qquad x + y + z = 1.$$

1. Show that the planes are not parallel and find the angle between them.
2. Find the equation of the line of intersection of the planes.

Solution

We need to be able to find the distance from a point to a plane. It turns out this is the same as finding the distance between parallel planes.

Example 1.2.14. Find the distance from the point $(-1,-2,3)$ to the plane $2x - 3y + z = 6$.

Solution

Example 1.2.15. Consider the line through the points $(1, 1, 0)$ and $(0, -2, 3)$. There are two planes that are perpendicular to $\mathcal{L}$ and are distance 2 units away from the point $(-3, 5, 1)$. Find the equations of these planes.

Solution

Definition 1.2.16. A line $\mathcal{L}$ is *parallel to a plane* $\mathcal{P}$ if a direction vector for $\mathcal{L}$ is orthogonal to a normal vector for $\mathcal{P}$.

We need to be able to find the distance from a line to a parallel plane.

Example 1.2.17. Find the distance from the line $\mathbf{r}(t) = \langle 5, 0, -1 \rangle + t \langle 1, -1, -2 \rangle$ to the plane $2x + 4y - z = 5$.

Solution

Example 1.2.18. Find an equation of the plane that is orthogonal to the plane $x - y + z = 3$ and contains the line $x - 3 = (y - 2)/3 = z/2$. Find the point of intersection of the line and the given plane if any.

Solution

1.2.4 Exercises

1. Find an equation of the plane passing through the point $(2, -3, 4)$ and normal to $\mathbf{n} = \langle -1, 2, 3 \rangle$.

2. Find an equation of the plane passing through the point $(2, -3, 4)$ and parallel to the plane $-2x + 4y - z = 17$.

3. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ the the planes described by $x + 2y + z = 5$ and $2x + y - z = 7$ respectively.

   (a) Find a parametric equation for the line of intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$.

   (b) Find an equation of the plane orthogonal to both $\mathcal{P}_1$ and $\mathcal{P}_2$ and passing through the point $(-1, 0, 3)$.

4. Find an equation of the plane that passes through the point $(3, 0, -2)$ and is parallel to the vectors $\langle 1, -3, 1 \rangle$ and $\langle 4, 2, 0 \rangle$.

5. Let $\mathcal{P}$ be the plane passing through $(1, -1, 5)$ and containing the vectors $\langle 1, -3, 1 \rangle$ and $\langle 4, 2, 0 \rangle$. Find the distance from the point $(0, 1, 12)$ the $\mathcal{P}$.

6. Find an equation of the plane passing through the points $(2, -1, 4)$, $(1, 1, -1)$ and $(-4, 1, 1)$.

7. Let $\mathbf{v}$ be a vector in the xy-plane that makes an angle of $\frac{\pi}{3}$ with the positive x-axis. Let $\widehat{\mathbf{u}}$ be a unit vector which is normal to the plane $3x + 4z = 0$ and forms an acute angle with the positive z-axis. Find $\text{proj}_{\mathbf{v}}\, \widehat{\mathbf{u}}$.

8. Consider the three planes

$$2x - y + z = 1, \qquad 4x - 2y + 2z = 1, \qquad -x + 3y - z = 5.$$

   (a) Determine which of the planes are parallel and which are intersecting

   (b) Find the distance between the pair of parallel planes

   (c) Find the angle between any two of the intersecting planes

9. Find the distance from the point $(1, 2, 3)$ to the plane $-x + 3y - z = 5$.

10. Find an equation of the plane that is equidistant from the points $(0, 1, 2)$ and $(4, -7, -10)$.

1.3 Geometry of Curves

1.3.1 Limits, Continuity, Derivatives and Integration

Definition 1.3.1. A *vector valued function of one variable* is a function whose domain is an interval and whose range is a curve (in either two-dimensional or three-dimensional space). A vector valued function takes the form

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle \qquad a \leq t \leq b,$$

where $x(t)$, $y(t)$ and $z(t)$ are real valued functions (i.e. calc 1 functions). The functions $x(t)$, $y(t)$ and $z(t)$ are called the *component functions* or the *coordinate functions* of the vector-valued function $\mathbf{r}$. The interval $[a, b]$ is called the *domain* or the *parameter domain*.

Remark 1.3.2. The interval in definition 1.3.1 need not be closed or bounded, the definition is just given in that case for the sake of concreteness.

Example 1.3.3. Graph the vector valued function

$$\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle \qquad 0 \leq t \leq 2\pi.$$

This is the standard parameterization for a helix.

Solution

Remark 1.3.4. Even though a curve is embedded in (two-dimensional or) three-dimensional space, we consider the curve to be a one-dimensional object. The reason is because the domain (an interval) is one-dimensional.

Limits

Recall the definition of limit of a scalar valued function (i.e. the calc 1 limit)

Definition 1.3.5. We say $\lim_{t \to t_0} f(t) = L$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|t - t_0| < \delta$ guarantees $|f(t) - L| < \epsilon$.

Example 1.3.6. Compute $\lim_{t \to 2} f(t)$, where

$$f(t) = \frac{t^2 - t - 2}{t - 2}.$$

Solution

To refresh your calc 1 skills, lets do the following example.

Example 1.3.7. Compute

$$\lim_{t \to 0} \frac{1}{t} \int_0^t \frac{1}{\pi + \arctan(s^2)} \, ds.$$

Solution

Now we give the limit of a vector-valued function.

Definition 1.3.8. Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. We say $\lim_{t \to t_0} \mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle$ if the following limits simultaneously hold (each in the calc 1 sense)

$$\begin{aligned} \lim_{t \to t_0} x(t) &= x_0 \\ \lim_{t \to t_0} y(t) &= y_0 \\ \lim_{t \to t_0} z(t) &= z_0. \end{aligned}$$

Alternatively, $\lim_{t \to t_0} \mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle$ if

$$\lim_{t \to t_0} \|\mathbf{r}(t) - \langle x_0, y_0, z_0 \rangle\| = 0.$$

The definition of $\lim_{t \to \pm\infty} \mathbf{r}(t)$ is defined similarly.

Example 1.3.9. Compute

$$\lim_{t\to 0}\left\langle \frac{\sin t}{t}, \frac{1-\cos t}{t}, \frac{e^t-1}{t}\right\rangle.$$

Solution

Example 1.3.10. Compute

$$\lim_{t\to 0}\left\langle \frac{t^3-1}{t-1},\ \sqrt{t},\ t^2\sin\left(\frac{1}{t}\right)\right\rangle$$

Solution

Continuity

Recall the definition of continuity of a scalar valued function (i.e. continuity in the calc1 sense)

Definition 1.3.11. We say a real-valued function f is *continuous at* $t = t_0$ if

$$\lim_{t \to t_0} f(t) = f(t_0).$$

This actually says three things

1. t_0 is in f's domain. (i.e. $f(t_0)$ makes sense)
2. $\lim_{t \to t_0} f(t)$ exists
3. the values of $f(t_0)$ and $\lim_{t \to t_0} f(t)$ coincide.

Definition 1.3.12. A vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is continuous if each of its component functions is continuous in the calc 1 sense. Specifically, $\mathbf{r}$ is continuous at t_0 if

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0).$$

In terms of the component functions, this says that the following simultaneously hold (in the calc 1 sense)

$$\begin{aligned} \lim_{t \to t_0} x(t) &= x(t_0) \\ \lim_{t \to t_0} y(t) &= y(t_0) \\ \lim_{t \to t_0} z(t) &= z(t_0). \end{aligned}$$

Differentiablility

Recall the definition of derivative for a real-valued function

Definition 1.3.13. We say a scalar valued function f is *differentiable at* t if

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

exists. If the limit exists, its value is the derivative of f at t and is denoted by $f'(t)$. If f is differentiable at each point of its domain, we say f is differentiable.

Definition 1.3.14. A vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is differentiable at a point t_0 of its domain (specifically, t_0 must be an interior point of the domain of $\mathbf{r}$) if each of the scalar-valued components are differentiable at t_0 in the calc 1 sense i.e. each of the following exist

$$\begin{aligned} x'(t_0) &= \lim_{h\to 0}\frac{x(t_0+h)-x(t_0)}{h} \\ y'(t_0) &= \lim_{h\to 0}\frac{y(t_0+h)-y(t_0)}{h} \\ z'(t_0) &= \lim_{h\to 0}\frac{z(t_0+h)-z(t_0)}{h}. \end{aligned}$$

In this case, the derivative of $\mathbf{r}$ at t_0 is

$$\mathbf{r}'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0)\rangle .$$

Similarly, if the coordinate functions of $\mathbf{r}(t)$ have higher-order derivatives, we can compute higher-order derivatives of $\mathbf{r}(t)$. For example,

$$\mathbf{r}''(t) = \langle x''(t), y''(t), z''(t)\rangle .$$

Remark 1.3.15. If $\mathbf{r}(t)$ is a parameterization for a curve C in space, then $\mathbf{r}'(t)$ is a vector which is tangent to C. The direction of $\mathbf{r}'(t)$ coincides with the direction in which $\mathbf{r}(t)$ traces out C. This intuition can help when asked to find tangent lines and normal planes to parameterized curves.

Example 1.3.16. Let

$$\mathbf{r}(t) = \left\langle \tan t \ , \int_0^t (s^2+1)^{-3/2} \, ds \ , 1 \right\rangle , \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

1. Compute $\mathbf{r}'(t)$.
2. Find an equation of the line tangent to $\mathbf{r}(t)$ at $t = 0$.
3. Find an equation of the plane which is normal to $\mathbf{r}(t)$ at $t = 0$.

Solution

In many physical applications, a parameterization $\mathbf{r}(t)$ for a curve C represents the position of an object at time t. As time progresses, the particle moves along C. That is, C is the particle's trajectory. In this case, we have the following interpretations of $\mathbf{r}(t)$ and its derivatives

$$\begin{aligned}
\mathbf{r}(t) &= \text{(position of object at time } t) \\
\mathbf{r}'(t) &= \text{(velocity of object at time } t) \\
\|\mathbf{r}'(t)\| &= \text{(speed of object at time } t) \\
\mathbf{r}''(t) &= \text{(acceleration of object at time } t).
\end{aligned}$$

Definition 1.3.17. Let $\mathbf{r}(t)$ be a differentiable vector-valued function. The *tangent vector* to $\mathbf{r}(t)$ is $\mathbf{r}'(t)$. The *unit tangent vector*, denoted $\widehat{\mathbf{T}}(t)$ is

$$\widehat{\mathbf{T}}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

provided $\|\mathbf{r}'(t)\| \neq 0$.

Remark 1.3.18. Although a curve can admit multiple tangent vectors, when we say "the tangent vector", we mean the vector tangent to the curve whose direction agrees with the direction of traversal of the curve and whose length agrees with the speed of traversal of the curve. Similarly, when we say "the unit tangent vector", we are referring to the tangent vector of unit length whose direction agrees with the direction of parameterization of the curve.

Definition 1.3.19. We say a vector valued function $\mathbf{r}(t)$ is *smooth* if the function $\widehat{\mathbf{T}}(t)$ is continuous for all $a \leq t \leq b$. In particular, $\mathbf{r}(t)$ must be differentiable and $\mathbf{r}'(t) \neq \mathbf{0}$ for all t.

Example 1.3.20. Let
$$\mathbf{r}(t) = \left\langle t^3, 3t^2, 0 \right\rangle.$$

1. Sketch the graph of $\mathbf{r}(t)$ [Hint: find a relationship between the component functions $x(t)$ and $y(t)$.] Is this a graph you would describe as "smooth"?
2. Compute $\mathbf{r}'(t)$ and $\mathbf{r}'(0)$.

Solution

Remark 1.3.21. The previous example shows why we require $\mathbf{r}'(t) \neq \mathbf{0}$ in the definition of smooth curve.

Example 1.3.22. Find the unit tangent vector for $\mathbf{r}(t) = \langle t^2, 4t, \ln t\rangle$, $t > 0$.

Solution

Proposition 1.3.23. *Rules for differentiation Let $\mathbf{r}(t)$ and $\mathbf{R}(t)$ be differentiable vector-valued functions, let $f(t)$ be a differentiable scalar-valued function and let $\mathbf{c}$ be a constant vector.*

1. $\frac{d}{dt}\mathbf{c} = \mathbf{0}$
2. $\frac{d}{dt}(\mathbf{r}(t) + \mathbf{R}(t)) = \frac{d}{dt}\mathbf{r}(t) + \frac{d}{dt}\mathbf{R}(t)$.
3. $\frac{d}{dt}(f(t)\mathbf{r}(t)) = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$.
4. $\frac{d}{dt}\mathbf{r}(f(t)) = \mathbf{r}'(f(t))f'(t)$.
5. $\frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{R}(t)) = \mathbf{r}'(t) \cdot \mathbf{R}(t) + \mathbf{r}(t) \cdot \mathbf{R}'(t)$
6. $\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{R}(t)) = \mathbf{r}'(t) \times \mathbf{R}(t) + \mathbf{r}(t) \times \mathbf{R}'(t)$
   (Be careful of the order in the cross product.)

Integration of Vector-Valued Functions

We define the integral of a vector valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ by integrating each of the component functions

$$\int \mathbf{r}(t) = \int \langle x(t), y(t), z(t)\rangle = \left\langle \int x(t), \int y(t), \int z(t) \right\rangle$$

The definite integral is defined similarly.

Example 1.3.24. Suppose $\mathbf{r}(t)$ is a vector valued function whose derivative is

$$\mathbf{r}'(t) = \left\langle \sec^2 t, \sqrt{t}, \frac{1}{t} \right\rangle, \qquad t > 0$$

and such that $\mathbf{r}(\pi/4) = \langle 1, 0, 0\rangle$. Find $\mathbf{r}(t)$.

Solution

1.3.2 Exercises

1. Graph the curve whose parametrization is $\mathbf{r}(t) = \langle 0, \sin(2\pi t), \cos(2\pi t)\rangle$, for $0 \le t \le 1$.

2. Graph the curve parameterized by $\mathbf{r}(t) = \langle \cos(2\pi t), \sin(2\pi t), e^{-t}\rangle$ for $0 \le t < \infty$.

3. Evaluate $\lim_{t\to 0} \mathbf{r}(t)$ for
$$\mathbf{r}(t) = \left\langle -\frac{3t}{\sin t},\ \frac{1}{t}\int_0^t e^{s^2}\,ds,\ \frac{e^t - t - 1}{t}\right\rangle$$

4. Find all points of intersection of the plane $x + y = 0$ and the curve whose parameterization is $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ for $0 \le t \le 4\pi$.

5. Compute $\mathbf{r}'(t)$ if
$$\mathbf{r}(t) = \left\langle \frac{1}{1+t^2},\ \cos^2(t),\ \ln(1+t^2)\right\rangle.$$
Where (i.e. for which values of t) is $\mathbf{r}(t)$ smooth?

6. Find a vector tangent to $\mathbf{r}(t) = \langle 4e^{2t}, e^{-2t}, 2e^t\rangle$ at the point $(36, \frac{1}{9}, 6)$.

7. Suppose $\mathbf{r}(t) = \left\langle t, \sin t, e^{-t^2}\right\rangle$ and $f(t) = \sqrt{t}$. Compute $\frac{d}{dt}\mathbf{r}(f(t))$.

8. Suppose $\mathbf{u}(t) = \langle 1, t, t^2\rangle$ and $\mathbf{v}(t) = \langle t^2, -2t, 1\rangle$. Compute $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t))$.

9. An object moves in space according to the function $\mathbf{r}(t) = \langle 1 + 4t^2, t\rangle$ for $-\infty < t < \infty$.
   (a) Find all values of t for which the object's position vectors and velocity vectors are are parallel.
   (b) Find all values of t for which the object's position vectors and velocity vectors are orthogonal.
   (c) Let $\mathbf{v}(t)$ and $\mathbf{a}(t)$ denote the velocity and the acceleration of the object respectively. Compute both $\mathrm{proj}_{\mathbf{v}}\,\mathbf{a}$ and $\mathrm{orth}_{\mathbf{v}}\,\mathbf{a}$ at $t = 1$. Are the velocity and acceleration vectors orthogonal?

10. Find an equation of the line tangent to $\mathbf{r}(t) = \left\langle 2\sin t, \sin\frac{t}{2}, 3\cos t\right\rangle$ at $t = \pi$.

11. Find an equation of the plane which is normal to $\mathbf{r}(t) = \langle \sin t, \cos t, e^{-t}\rangle$ at $t = 0$. [Hint: The normal vector for the plane should be parallel to the curve's tangent vector at $t = 0$].

12. Compute the indefinite integral $\int \mathbf{r}(t)\, dt$ of the vector valued function

$$\mathbf{r}(\mathbf{t}) = \left\langle \frac{t}{\sqrt{t^2+1}},\ \tan t,\ t^2+3 \right\rangle.$$

13. Compute $\int_1^4 \mathbf{r}(t)\, dt$ if $\mathbf{r}(t) = \left\langle \frac{1}{t}, e^{2t}, \frac{1}{\sqrt{t}} \right\rangle$.

14. An object moves through space with velocity

$$\mathbf{v}(t) = \left\langle \frac{1}{1+t^2}, 1, e^{-2t} \right\rangle, \qquad t \geq 0.$$

If the initial position of the object is $(0, 0, 1)$, find the position of the object at time t.

1.3.3 Arclength and Natural Parameterization

Suppose a curve C is parameterized by a differentiable vector valued function $\mathbf{r}(t)$ for $a \leq t \leq b$. We can approximate C with a sequence of segments whose endpoints are on C. By adding the lengths of the segments we get an approximation for the length of C. By using smaller (and hence more) segments in this approximation procedure we can get better approximations for the length of C. By letting the length of the approximating segments go to zero, we get the length of C.

Example 1.3.25. Derive an expression for the arc length of a curve C parameterized by a differentiable function $\mathbf{r}(t)$ if $\mathbf{r}(t)$ has the special form $\mathbf{r}(t) = \langle t, y(t) \rangle$.

Proposition 1.3.26. *Let $\mathbf{r}(t)$ be a parameterization for a curve C, where $a \leq t \leq b$. The arc length of C (also called the length of C) is denoted by s and is given by*

$$s = \int_a^b \|\mathbf{r}'(t)\| \ dt. \tag{1.16}$$

Remark 1.3.27. If $\mathbf{r}(t)$ is the position of an object at time t, then the arc length of the curve C whose parameterization is $\mathbf{r}(t)$ for $a \leq t \leq b$ represents the total distance traveled by the object between times $t = a$ and $t = b$. If $\mathbf{v}(t)$ is the velocity of the object, then equation (1.16) may be expressed in the equivalent form

$$s = \int_a^b \|\mathbf{v}(t)\| \ dt.$$

Here's a quick example to convince you that this formula actually gives you the length of a curve.

Example 1.3.28. Consider the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \leq t \leq 2\pi$.

1. Compute the length of the helix using purely geometrical reasoning.

2. Compute the length of the helix using the arc length formula.

Solution

Example 1.3.29. Compute the length of the curve C parameterized by $\mathbf{r}(t) = \langle 3\cos t, 4\cos t, 5\sin t\rangle$ for $0 \leq t \leq 2\pi$.

Solution

Example 1.3.30. The position of an object at time t is given by

$$\mathbf{r}(t) = \left\langle \frac{2}{3}\cos^3 t, \frac{2}{3}\sin^3 t \right\rangle.$$

Compute the length of the objects trajectory.

Solution

Reparameterization and Natural Parameterization

Suppose a curve C is parameterized by $\mathbf{r}(t)$ for $a \leq t \leq b$. Suppose t is a function of another real variable u, so $t = f(u)$ for $\alpha \leq u \leq \beta$, where f' exists, is continuous and is positive for $\alpha < u < \beta$. Then we can get another parameterization for C via the formula

$$\mathbf{R}(u) = \mathbf{r}(f(u)) \qquad \alpha \leq u \leq \beta.$$

Example 1.3.31. Let C be a single turn of a helix with parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \leq t \leq 2\pi$.

1. Find a constant-speed parameterization of C with parameter domain $[0, 1]$.
2. Find a parameterization of C with parameter domain $[0, 1]$ whose speed of traversal is non-constant.

Solution

Definition 1.3.32. Suppose C is a smooth curve in space. The *natural parameterization* or *arc length parameterizaion* for C is the parameterization $\mathbf{r}(s)$ such that

$$\|\mathbf{r}'(s)\| = 1 \qquad \text{for all } 0 \leq s \leq L, \tag{1.17}$$

where L is the length of C. In this case s is called the *natural parameter* or the *arc length parameter*.

Remark 1.3.33. Equation (1.17) says that the natural parameterizaion for a curve C traverses C with constant speed 1.

Example 1.3.34. Although the parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ for $0 \leq t \leq 2\pi$ is a standard parameterization for a single turn of a helix, it is not the natural parameterization. Explain why and find the natural parameterization.

Solution

In general, finding the natural parameterization of a curve with a given parameterization is more difficult than in the previous example.

Definition 1.3.35. Let C be a curve in space parameterized by a continuously differentiable function $\mathbf{r}(t)$ and suppose $\|\mathbf{r}'(t)\| > 0$ for all $a \leq t \leq b$. Then the natural parameter (also called the arc length parameter) is

$$s = \int_a^t \|\mathbf{r}'(u)\| \; du, \qquad a \leq t \leq b. \tag{1.18}$$

Observe that the $0 \leq s \leq L$, where L is the length of C (and is given by $s(b)$).

Example 1.3.36. Let C be the curve parameterized by $\mathbf{r}(t) = \langle 2t^2, t^2 4t^2 \rangle$ for $1 \leq t \leq 4$.

1. Determine whether C is parameterized with the arc length (i.e. the natural) parameterization. If not, find the natural parameterization.

2. In either case, find the length of C using the natural parameterization.

Solution

Remark 1.3.37. If a curve C is parameterized by $\mathbf{r}(t)$ for $a \le t \le b$ and if $\|\mathbf{r}'(t)\| \neq 0$ for all $t \in [a, b]$, then the equation

$$s(t) = \int_a^t \|\mathbf{r}'(u)\| \; du$$

can be solved for t (to give t as a formula involving s). In this case, we can use the method of example 1.3.36 to find the natural parameterization.

1.3.4 Exercises

1. Let C be the curve parameterized by $\mathbf{r}(t) = \langle \ln(\cos t), t \rangle$ for $0 \le t \le \pi/4$.
   (a) Show that the given parameterization of C is not the natural parameterization.
   (b) Evaluate an appropriate integral to find a formula for the natural parameter s in terms of the given parameter t.
2. The trajectory of an object is given by $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$ for $0 \le t \le 1$.
   (a) Show that the given parameterization is not the natural parameterization.
   (b) Find the natural parameterization.
   (c) Find the total distance traveled by the object.
3. Find the arc length parameterization of the line

   $$x(t) = 2 + 4t \qquad y(t) = 1 - 12t \qquad z(t) = 3 + 3t$$

   in terms of the arc length s measured from the initial point $(2, 1, 3)$.
4. Let C be the curve parameterized by

   $$\mathbf{r}(t) = \langle \ln(\cos t), \ln(\cos t), 2t \rangle \qquad 0 \le t \le \frac{\pi}{4}.$$

   (a) Show that t is not the natural parameter.
   (b) Evaluate an appropriate integral to obtain a formula for the natural parameter s in terms of t.
   (c) Use the formula you obtained in the previous part to compute the length of C.
5. Let C be the curve parameterized by

   $$\mathbf{r}(t) = \left\langle \sin^2 t, \cos^2 t, \frac{2}{3} \sin^3 t \right\rangle \qquad 0 \le t \le \frac{\pi}{2}.$$

(a) Show that t is not the natural parameter.

(b) Evaluate an appropriate integral to obtain a formula for the natural parameter s in terms of t.

(c) Use the formula you obtained in the previous part to compute the length of C.

1.3.5 Curvature and the $\widehat{\mathbf{T}}$-$\widehat{\mathbf{N}}$-$\widehat{\mathbf{B}}$ Frame

Definition 1.3.38. Let C be a smooth curve parameterized by $\mathbf{r}(s)$ for $0 \leq s \leq L$, where s is the arc length (natural) parameter. The *curvature* of C is

$$\kappa(s) = \left\| \frac{d\widehat{\mathbf{T}}}{ds} \right\|. \tag{1.19}$$

Remark 1.3.39. In many cases, you will be given a non-natural parameterization $\mathbf{r}(t)$ for a curve C and asked to find the curvature κ. In this case, you could first find the natural parameterization, then find κ using formula (1.19). A computationally simpler approach is the following. According to equation (1.18) and the fundamental theorem of calculus,

$$\frac{ds}{dt} = \left\| \mathbf{r}'(t) \right\|.$$

Therefore,

$$\left\| \frac{d\widehat{\mathbf{T}}}{dt} \right\| = \left\| \frac{d\widehat{\mathbf{T}}}{ds} \frac{ds}{dt} \right\| = \kappa \left\| \mathbf{r}'(t) \right\|,$$

so

$$\kappa(t) = \frac{1}{\left\| \mathbf{r}'(t) \right\|} \left\| \frac{d\widehat{\mathbf{T}}}{dt} \right\|. \tag{1.20}$$

Notice that the derivative is taken with respect to the (given) parameter t. Also, if the given parameter happens to be the natural parameter, then $\left\| \mathbf{r}'(t) \right\| = 1$ and equation (1.20) reduces to equation (1.19)

Example 1.3.40. Compute the curvature of the curve parameterized by $\mathbf{r}(t) = \langle 1, 0, 4 \rangle + t \langle 5, -1, 2 \rangle$ for $-\infty < t < \infty$. Does your answer make sense?

Solution

Example 1.3.41. Compute the curvature of a circle of radius R.

Solution

For more complicated curves, a computationally convenient formula for the curvature is given by the following proposition.

Proposition 1.3.42. *Let C be a curve with smooth parameterization $\mathbf{r}(t)$ for $a \leq t \leq b$. The curvature is given by*

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\|\mathbf{v}(t)\|^3}$$

Definition 1.3.43 (Principal Unit Normal Vector). Let C be a smooth curve with natural parameter s. The *principal unit normal vector* $\widehat{\mathbf{N}}$ is given by

$$\widehat{\mathbf{N}} = \frac{\frac{d\widehat{\mathbf{T}}}{ds}}{\left\|\frac{d\widehat{\mathbf{T}}}{ds}\right\|} = \frac{1}{\kappa}\frac{d\widehat{\mathbf{T}}}{ds} \tag{1.21}$$

$\widehat{\mathbf{N}}$ is defined whenever $d\widehat{\mathbf{T}}/ds \neq 0$.

Suppose $\mathbf{r}(t)$ is a smooth parameterization for C. Since $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$, we get the following formula for $\widehat{\mathbf{N}}$ in terms of the given parameter t

$$\widehat{\mathbf{N}} = \frac{\frac{d\widehat{\mathbf{T}}}{dt}\Big/\frac{ds}{dt}}{\left\|\frac{d\widehat{\mathbf{T}}}{dt}\Big/\frac{ds}{dt}\right\|} = \frac{\frac{d\widehat{\mathbf{T}}}{dt}\Big/\|\mathbf{r}'(t)\|}{\left\|\frac{d\widehat{\mathbf{T}}}{dt}\Big/\|\mathbf{r}'(t)\|\right\|} = \frac{\frac{d\widehat{\mathbf{T}}}{dt}}{\left\|\frac{d\widehat{\mathbf{T}}}{dt}\right\|}$$

Example 1.3.44. Let $\mathbf{r}(t) = \langle 3t, 2t^2 \rangle$.

1. Compute both $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$.
2. Compute $\widehat{\mathbf{T}} \cdot \widehat{\mathbf{N}}$. What do you conclude?

Solution

Proposition 1.3.45. *Let C be a smoothly parameterized curve with unit tangent and principal normal vectors $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$ respectively.*

1. $\widehat{\mathbf{T}} \cdot \widehat{\mathbf{N}} = 0$ *(i.e. $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$ are orthogonal).*
2. *The plane determined by $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$ contains both $\mathbf{r}'$ and $\mathbf{r}''$.*
3. *The principal unit normal vector $\widehat{\mathbf{N}}$ points "into the direction of the curve".*

Here is a picture:

Example 1.3.46. Let C be the curve parameterized by $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$. Find the equation of the plane that is tangent to C at $t = \pi/4$ and which contains the vectors $\mathbf{r}'(\pi/4)$ and $\mathbf{r}''(\pi/4)$.

Solution

Decomposition of Accelerateion

If $\mathbf{r}(t)$ is the position of an object at time t, then $\mathbf{r}''(t) = \mathbf{a}(t)$ is the acceleration of the object at time t. Since $\mathbf{a}(t)$ is in the plane determined by $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$, and since $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$ are orthogonal unit vectors, we can decompose $\mathbf{a}(t)$ relative to $\widehat{\mathbf{T}}$ and $\mathbf{N}$ in a simple way using projections.

Definition 1.3.47. The *tangential component of acceleration*, denoted a_T, is given by

$$a_T = \mathbf{a} \cdot \widehat{\mathbf{T}} = \pm \left\| \operatorname{proj}_{\widehat{\mathbf{T}}} \mathbf{a}. \right\|$$

The *normal component of acceleration*, denoted a_N, is given by

$$a_N = \mathbf{a} \cdot \widehat{\mathbf{N}} = \pm \left\| \operatorname{proj}_{\widehat{\mathbf{N}}} \mathbf{a} \right\|.$$

In particular, $\mathbf{a}(t)$ admits the decomposition

$$\mathbf{a}(t) = a_T \widehat{\mathbf{T}} + a_N \widehat{\mathbf{N}} = (\mathbf{a} \cdot \widehat{\mathbf{T}})\widehat{\mathbf{T}} + (\mathbf{a} \cdot \widehat{\mathbf{N}})\widehat{\mathbf{N}}.$$

Proposition 1.3.48. *The scalar tangential component a_T and the scalar normal component a_N of acceleration are given by*

$$a_T = \frac{d}{dt}\left(\|\mathbf{r}'(t)\|\right) \qquad \textit{and} \qquad a_N = \kappa \|\mathbf{r}'(t)\|^2 = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Proof. ∎

Example 1.3.49. Let

$$\mathbf{r}(t) = \left\langle 4t^2, \frac{1}{2}(t-1)^2, t \right\rangle \qquad 0 \leq t \leq 1$$

be the position of an object at time t, Find the tangential and normal components of the objects acceleration at time t.

Solution

Unit Binormal Vector and Torsion

Definition 1.3.50. Let C be a smoothly parameterized curve with with unit tangent vector $\widehat{\mathbf{T}}$ and principal unit normal vector $\widehat{\mathbf{N}}$. The *unit binormal vector* $\widehat{\mathbf{B}}(t)$ is given by

$$\widehat{\mathbf{B}}(t) = \widehat{\mathbf{T}}(t) \times \widehat{\mathbf{N}}(t).$$

Example 1.3.51. Let $R > 0$ and let C be the curve parameterized by $\mathbf{r}(t) = \langle R\cos t, R\sin t, 0\rangle$ for $0 \le t \le 2\pi$.

1. Sketch C on a three dimensional axis.
2. Compute both $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$ at $t = \pi/4$. Sketch these two vectors on the sketch you made in the previous part.
3. Find and sketch $\widehat{\mathbf{B}}$. [Hint: given the previous parts, it is possible to find $\widehat{\mathbf{B}}$ without any further computation.]

Solution

Proposition 1.3.52. *Let C be a curve with smooth parameterization $\mathbf{r}(t)$. The unit binormal vector $\widehat{\mathbf{B}}$ satisfies*

$$\widehat{\mathbf{B}} = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|} \tag{1.22}$$

Proposition 1.3.53. *Let C be a smooth curve in space and let $\widehat{\mathbf{T}}$, $\widehat{\mathbf{N}}$ and $\widehat{\mathbf{B}}$ be the unit tangent, unit normal and unit binormal vectors respectively. Then $\widehat{\mathbf{T}}$, $\widehat{\mathbf{N}}$ and $\widehat{\mathbf{B}}$ (in that order) forms a right-handed coordinate system called the $\widehat{\mathbf{T}} - \widehat{\mathbf{N}} - \widehat{\mathbf{B}}$ frame.*

Remark 1.3.54. In many cases computing $\widehat{\mathbf{N}}$ is tedious. However, in view of proposition 1.3.53, we have

$$\widehat{\mathbf{N}} = \widehat{\mathbf{B}} \times \widehat{\mathbf{T}}.$$

Since $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{B}}$ are easy to compute (e.g. using equation (1.22) to compute $\widehat{\mathbf{B}}$), this gives a less tedious way to compute $\widehat{\mathbf{N}}$.

Definition 1.3.55. Let C be a curve with smooth parameterization $\mathbf{r}(t)$ and let t_0 be in the parameter domain.

1. The plane containing $\mathbf{r}(t_0)$ and normal to $\widehat{\mathbf{B}}(t_0)$ is called the *osculating plane* to C at t_0. The osculating plane at t_0 contains both $\mathbf{r}'(t_0)$ and $\mathbf{r}''(t_0)$.

2. The plane containing $\mathbf{r}(t_0)$ and normal to $\widehat{\mathbf{T}}(t_0)$ is called the *normal plane* to C at t_0.

3. The plane containing $\mathbf{r}(t_0)$ and normal to $\widehat{\mathbf{N}}(t_0)$ is called the *rectifying plane* to C at t_0.

Remark 1.3.56. We actually found an equation of an osculating plane in example 1.3.46.

Example 1.3.57. Find equations of the osculating, normal and rectifying planes to the curve parameterized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $(1, 1, 1)$.

Solution

Definition 1.3.58. Let C be a smoothly parameterized curve with unit normal and binormal vectors $\widehat{\mathbf{N}}$ and $\widehat{\mathbf{B}}$ respectively. The *torsion* τ of C is

$$\tau = -\frac{d\widehat{\mathbf{B}}}{ds} \cdot \widehat{\mathbf{N}}(s),$$

where s is the arc length parameter.

Proposition 1.3.59. *If C is a smooth planar curve then the torsion of C is zero.*

Proposition 1.3.60. *Let C be a smooth curve parameterized by $\mathbf{r}(t)$. Then the torsion is given by*

$$\tau = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}.$$

Example 1.3.61. Let C be parameterized by $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 1 - 2\cos t\rangle$ for $0 \leq t \leq 2\pi$. Determine whether C is planar.

1.3.6 Exercises

1. Consider the curve $y = x^2$ in the xy-plane for $-\infty < x < \infty$.
   (a) Sketch and find a parameterization for C.
   (b) According to your sketch, which point on C is curved the most?
   (c) Compute the curvature κ of C. Does the formula you obtained coincide with your answer to the previous question?

2. Let $\mathbf{r}(t) = \langle 3t, 2t^2, 0\rangle$. In example 1.3.44 we computed
$$\begin{aligned} \mathbf{r}'(t) &= \langle 3, 4t, 0\rangle \\ \mathbf{r}''(t) &= \langle 0, 4, 0\rangle \\ \widehat{\mathbf{T}}(t) &= (9+4t^2)^{-\frac{1}{2}}\langle 3, 4t, 0\rangle \\ \widehat{\mathbf{T}}'(t) &= (9+4t^2)^{-\frac{3}{2}}\langle -12t, 9, 0\rangle . \end{aligned}$$
   (a) Compute $\mathbf{r}'(1)$, $\mathbf{r}''(1)$, $\widehat{\mathbf{T}}(1)$ and $\widehat{\mathbf{N}}(1)$ [note that $\widehat{\mathbf{N}}$ is not given above].
   (b) Find an equation of the plane containing both $\mathbf{r}'(1)$ and $\mathbf{r}''(1)$ which passes through the point $(3, 2, 0)$.
   (c) Find and equation of the plane containing both $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$ which passes through $(3, 2, 0)$.
   (d) What can you say regarding the planes whose equations you just found? Why is this true?

3. Let $\mathbf{r}(t) = \left\langle \frac{t^2}{2}, 4-3t, 1\right\rangle$ for $-\infty < t < \infty$.
   (a) Compute the unit tangent vector $\widehat{\mathbf{T}}$.
   (b) Compute the principal unit normal vector $\widehat{\mathbf{N}}$.
   (c) Compute the binormal vector $\widehat{\mathbf{B}}$.
   (d) Perform some quick computations to show that $\widehat{\mathbf{T}}$, $\widehat{\mathbf{N}}$ and $\widehat{\mathbf{B}}$ are pairwise orthogonal (this is how you can check your answer).

4. Let
$$\mathbf{r}(t) = \left\langle \int_0^t \cos\left(\frac{\pi^2 t}{2}\right)\,dt, \int_0^t \sin\left(\frac{\pi^2 t}{2}\right)\,dt \right\rangle, \qquad \text{for } t > 0.$$
   Find both $\widehat{\mathbf{T}}(t)$ and $\kappa(t)$.

5. Find the principal unit normal vector $\widehat{\mathbf{N}}$ to the curve parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t, 2+t\rangle$ at the point $(1, 0, 2)$.

6. Consider the curve C parameterized by $\mathbf{r}(t) = \langle 3\sin t, 3\cos t\rangle$. Find $\widehat{\mathbf{N}}(t)$, the principal unit normal vector to C.

7. Let C be the curve parameterized by $\mathbf{r}(t) = \left\langle \sqrt{3}\sin t, \sin t, 2\cos t\right\rangle$ for $\leq t \leq$. Find the unit tangent vector $\widehat{\mathbf{T}}(t)$ and the curvature $\kappa(t)$

8. Let C be the curve parameterized by $\mathbf{r}(t) = \langle t^2, t, t\rangle$ for $-\infty < t < \infty$.

   (a) Find $\widehat{\mathbf{T}}$, $\widehat{\mathbf{N}}$ and $\widehat{\mathbf{B}}$ at $t = 2$.

   (b) Find an equation of the osculating plane at $t = 2$.

   (c) Find an equation of the normal plane at $t = 2$.

   (d) Find an equation of the rectifying plane at $t = 2$.

9. Let C be the curve parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ for $0 \leq t \leq 2\pi$. Find an equation of the osculating plane to C at the point $(0, 1, \frac{\pi}{2})$.

10. Let C be the curve with parameterization

$$\mathbf{r}(t) = \left\langle \cos t,\ 2\sin t,\ \sqrt{35 - 3\sin^2 t}\right\rangle \qquad 0 \leq t \leq 2\pi.$$

    Determine whether C is planar.

1.4 Exam 1 Review

Problem 1.1. The following are the equations of three distinct planes

$$2x - 3y + 6z = 1, \qquad 3x - 2y - 6z = 2, \qquad \text{and} \qquad -4x + 6y - 12z = 8.$$

1. Two of the planes are parallel. Determine which two and find the distance between them.

2. Find the angle $\theta \in (0, \pi/2]$ between any two of the non-parallel planes.

Problem 1.2. The following equations represent three distinct lines.

$$\frac{x-1}{3} = \frac{y}{4} = \frac{z+5}{12}, \qquad x = 3t+1, y = 4t-2, z = 12t, \qquad \mathbf{r}(t) = \langle 2t+1, 3t-2, 6t\rangle.$$

Determine which of the lines are parallel and find the distance D between them.

Problem 1.3. The motion of an object is described by the function

$$\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle \qquad \text{for } \ t \geq 0.$$

1. Find both the velocity $\mathbf{v}(t)$ and the speed $\upsilon(t)$. Compute both $\lim_{t\to\infty}\mathbf{r}(t)$ and $\lim_{t\to\infty}\upsilon(t)$.

2. Find the arc length L of the objects trajectory from time $t=0$ to time $t=1$. Express the length of the objects trajectory as a function of t for $t \geq 0$ (i.e. find $L(t)$). Compute $\lim_{t\to\infty} L(t)$ and interpret what the limit represents.

Problem 1.4. A curve C is defined by the parametric equations

$$x = t, \quad y = t^2, \quad z = t^4 \qquad -\infty < t < \infty.$$

1. Find an equation of the osculating plane at the point $(1,1,1) \in C$.

2. Compute each of the following at $(1,1,1)$: the curvature κ, the unit tangent vector $\widehat{\mathbf{T}}$ and the binormal vector $\widehat{\mathbf{B}}$.

3. Calculate the torsion τ at the point $(1,1,1)$.

4. Is the curve C smooth? Is the curve planar? Justify your answers.

Chapter 2

Differentiation

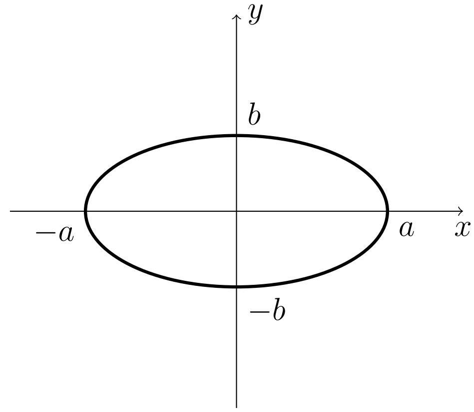
2.1 Functions of Several Variables

Let us begin with a precalculus review of ellipses and hyperbolas.

Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

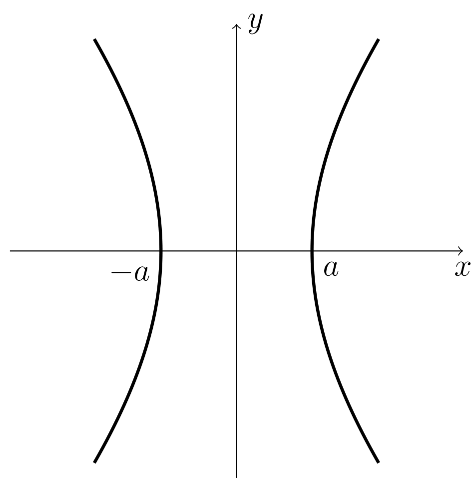
- x-intercepts: $\pm a$.
- y-intercepts: $\pm b$.
- Special case: $a = b$ gives a circle of radius a.



Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

- x-intercepts: $\pm a$.
- y-intercepts: none.
- As $|x| \to \infty$, graph approaches the lines $y = \pm \frac{b}{a}x$.

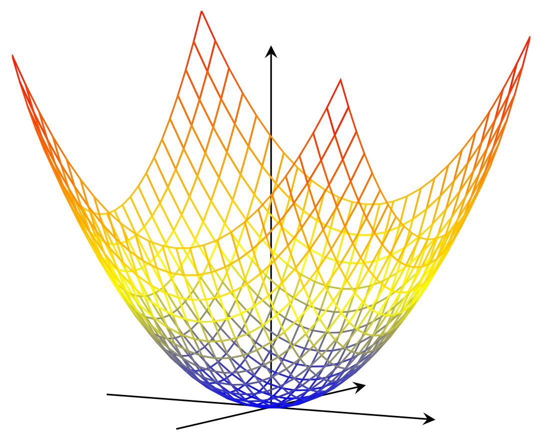


Now let us list some shapes that will be used frequently in calculus 3.

Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

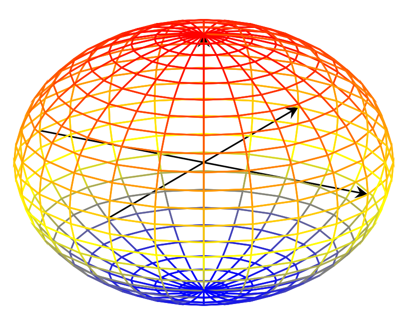
- Domain: R^2
- Range: $[0, \infty]$
- x-slices parabolas
- y-slices parabolas
- z-slices ellipses



Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

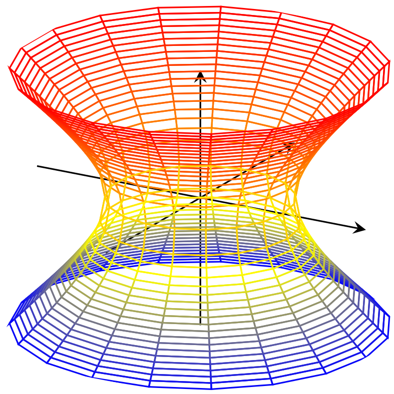
- Special case: $a = b = c$ gives a sphere of radius a.
- All slices are ellipses.



Hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

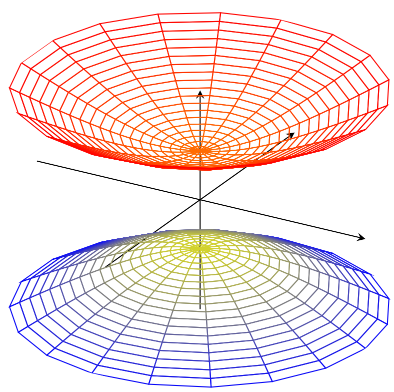
- x-slices: hyperbolas
- y-slices: hyperbolas
- z-slices ellipses



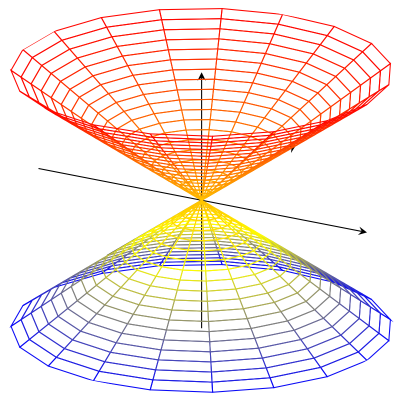
Hyperboloid of two sheets

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

- x-slices hyperbolas
- y-slices hyperbolas
- z-slices ellipses
(only exist for $|z| > c$)



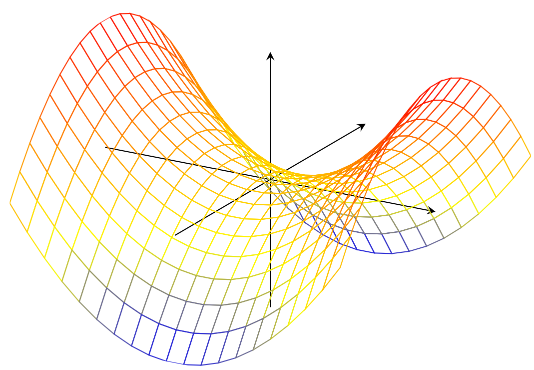
Elliptic Cone



$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- x-slices: intersecting lines if $x = 0$, hyperbolas if $x \neq 0$.
- y-slices: intersecting lines if $y = 0$, hyperbolas if $y \neq 0$.
- z-slices: ellipses if $z \neq 0$.

Hyperbolic Paraboloid



$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

- x-slices: parabolas.
- y-slices: parabolas.
- z-slices: hyperbola if $z \neq 0$.

Definition 2.1.1. Let $D \subset \mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a function with domain D. If $z_0 \in \text{Range}(f)$, the *level set* of f corresponding to level z_0 is

$$\{(x, y) \in D : f(x, y) = z_0\}.$$

Remark 2.1.2. The level set of a function is in the domain space of the function.

Example 2.1.3. Find and sketch the level sets of $z = f(x, y) = 4 - x^2 - y^2$ corresponding to levels $z = 0, 1, 2, 3, 4$.

Solution

Remark 2.1.4. As in example 2.1.3, often times the level sets of a function will have a "nice" parameterization. For example, in example 2.1.3 the level set corresponding to $z = 0$ has parameterization

$$\mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle \qquad 0 \le t \le 2\pi.$$

Notice that this parameterizes a curve in the xy-plane.

Definition 2.1.5. If $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ is a function with domain D, the collection of level curves of f is called a *contour map* for f (and it lives in f's domain space).

Remark 2.1.6. Loosely speaking, if $E \subset \mathbb{R}^3$ and $f : E \to \mathbb{R}$ is a function with domain E, then the level sets of f are surfaces.

Example 2.1.7. let

$$w = f(x, y, z) = \sqrt{16 - x^2 - y^2 - z^2}.$$

1. Find the range of f.
2. Compute and sketch the level surfaces of f corresponding to $w = 0, 2, 4$.

Solution

2.1.1 Exercises

1. Find the domain D of $f(x, y) = \frac{1}{x}\sqrt{x^2 + y^2 - 16}$. Plot D in the xy-plane.
2. Find and sketch the domain of each of the following functions.
   (a) $f(x, y) = \arcsin(1 + x + y)$.
   (b) $f(x, y) = \arccos(2y/x)$.
   (c) $f(x, y) = \ln(x^2 y - 9)$.
3. Graph a few (easy to compute) level curves for the following functions on the indicated domains
   (a) $z = f(x, y) = \sqrt{25 - x^2 - y^2}$ on $\{(x, y) : x^2 + y^2 \leq 25\}$.
   (b) $z = f(x, y) = x - y^2$ on $0 \leq x \leq 4$, $-2 \leq y \leq 2$.
   (c) $z = f(x, y) = \sqrt{x + 4y^2}$ on $-8 \leq x \leq 8$, $-8 \leq y \leq 8$.
   (d) $z = f(x, y) = e^{-x^2 - 2y^2}$ on $-2 \leq x \leq 2$, $-2 \leq y \leq 2$.
4. Let $f(x, y) = x^2 + y^2$.
   (a) Sketch the level sets of f corresponding to the levels $z = 0$, $z = 1$, $z = 2$ $z = 3$ and $z = 4$.
   (b) Use the two-dimensional picture you made in part (a) to sketch the graph of f.
5. Let $f(x, y) = \sqrt{16 - 4x^2 - y^2}$.
   (a) Compute the domain of f.
   (b) Compute the range of f.
   (c) Sketch the level sets of f corresponding to the levels $z = 0$, $z = 1$ and $z = 4$.
   (d) What happens if you try to compute the level set of f corresponding to $z = 5$? How does this relate to a previous part of this problem?
6. Sketch the level surfaces of the function $w = f(x, y, z) = \sqrt{z - x^2 - 2y^2}$ corresponding to the levels $w = 1$ and $w = 2$.
7. Sketch the level surfaces for
$$w = f(x, y, z) = \frac{1}{x^2 + y^2 + z^2}$$
corresponding to the levels $w = 1, 2, 3, 4$. What happens if you try to sketch the level set corresponding to $w = 0$? Why does this happen?

2.2 Limits and Continuity

Definition 2.2.1. Let $\mathbf{r}_0 = (x_0, y_0)$ be a specific point in $\mathbb{R}^2$ and let $\mathbf{r} = (x, y)$ be a generic point in $\mathbb{R}^2$. The *ball of radius* δ *and center* $\mathbf{r}_0$, denoted $B_\delta(\mathbf{r}_0)$ is the set of points whose distance to $\mathbf{r}_0$ is less than δ. In symbols,

$$\begin{aligned} B_\delta(\mathbf{r}_0) &= \{\mathbf{r} \in \mathbb{R}^2 : \|\mathbf{r} - \mathbf{r}_0\| < \delta\} \\ &= \{(x, y) \in \mathbb{R}^2 : \|(x, y) - (x_0, y_0)\| < \delta\} \\ &= \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\}. \end{aligned}$$

One dimensional balls and higher-dimensional balls are defined similarly.

Definition 2.2.2. Let $D \subset \mathbb{R}^2$.

1. A point $\mathbf{r}_0 \in D$ is called an *interior point of* D if there is $\delta > 0$ such that $B_\delta(\mathbf{r}_0) \subset D$. That is, $\mathbf{r}_0$ is an interior point if we can fit a (small) ball whose center is $\mathbf{r}_0$ inside of D.

2. A point $\mathbf{r}_0$ is called a *boundary point of* D if for all $\delta > 0$, $B_\delta(\mathbf{r}_0)$ intersects both D and $\mathbb{R}^2 \setminus D$. That is, $\mathbf{r}_0$ is a boundary point of D if every ball with center $\mathbf{r}_0$ contains both points in D and points not in D. The boundary of D will be denoted ∂D.

3. The *closure* of D, denoted $\overline{D}$ is $D \cup \partial D$. That is, $\overline{D}$ consists of the points of D together with all points of the boundary of D.

Here is a picture:

Remark 2.2.3.

1. The definitions of interior point and boundary point are similar if we replace $\mathbb{R}^2$ with $\mathbb{R}^n$.

2. If $\mathbf{r}_0$ is an interior point of D, then $\mathbf{r}_0$ must be inside of D.

3. if $\mathbf{r}_0$ is a boundary point of D, then $\mathbf{r}_0$ may be either inside D or outside of D.

Definition 2.2.4. Let $D \subset \mathbb{R}^2$ and let $\mathbf{r}_0 \in \overline{D}$. If $f : D \to \mathbb{R}$ is a function with domain D, then $\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r}) = L$ if for all $\epsilon > 0$, there is $\delta > 0$ (depending on ϵ) such that $\mathbf{r} \in B_\delta(\mathbf{r}_0)$ guarantees $|f(\mathbf{r}) - L| < \epsilon$.

Theorem 2.2.5 (Squeeze Theorem). *Suppose f, g and h are functions defined near $\mathbf{r}_0$ that satisfy*

$$g(\mathbf{r}) \le f(\mathbf{r}) \le h(\mathbf{r})$$

for all $\mathbf{r}$ near $\mathbf{r}_0$ (but possibly excluding $\mathbf{r}_0$). If both $\lim_{\mathbf{r}\to\mathbf{r}_0} g(\mathbf{r}) = L$ and $\lim_{\mathbf{r}\to\mathbf{r}_0} h(\mathbf{r}) = L$, then $\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r}) = L$.

Example 2.2.6. Show that

$$\lim_{\mathbf{r}\to\mathbf{0}} (x^2 + 2y^2 + 3z^2) \sin\left(\frac{1}{xyz}\right) = 0.$$

Solution

Remark 2.2.7. A common application of the squeeze theorem is as follows. Suppose a function f with domain D is given and that $\mathbf{r}_0 \in \overline{D}$. Suppose L is a candidate for $\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r})$. In theorem 2.2.5, replace g by 0, replace f by $|f(\mathbf{r}) - L|$ and replace h with a function involving $\|\mathbf{r} - \mathbf{r}_0\|$ whose limit is 0 as $\mathbf{r} \to \mathbf{r}_0$.

Here are some useful inequalities to make the method of remark 2.2.7 easier.

1. $|x| \leq \|\mathbf{r}\|$, $|y| \leq \|\mathbf{r}\|$ and $|z| \leq \|\mathbf{r}\|$.
2. $|\cos(\cdot)| \leq 1$ and $|\sin(\cdot)| \leq 1$.
3. $|a + b| \leq |a| + |b|$.
4. $|ab| = |a|\,|b|$.
5. $|a/b| = |a|\,/\,|b|$ provided $b \neq 0$.

Example 2.2.8. Use the method of remark 2.2.7 to show

$$\lim_{\mathbf{r}\to\mathbf{0}} (x^2 + 2y^2 + 3z^2) \sin\left(\frac{1}{xyz}\right) = 0.$$

Solution

Proposition 2.2.9. *Suppose $D \subset \mathbb{R}^2$ and $\mathbf{r}_0$ is a boundary point of D which is not a point in D. Suppose $f : D \to \mathbb{R}$ is a function whose domain is D and h is a function satisfying*

$$h(\mathbf{r}) = f(\mathbf{r}) \qquad \textit{for all } \mathbf{r} \in D.$$

If $\lim_{\mathbf{r}\to\mathbf{r}_0} h(\mathbf{r}) = L$, then $\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r}) = L$,

Example 2.2.10. Use proposition 2.2.9 to compute $\lim_{\mathbf{r}\to(1,-2)} f(\mathbf{r})$, where

$$f(\mathbf{r}) = \frac{y^2 + 2xy}{y + 2x}.$$

Solution

Example 2.2.11. Determine whether

$$\lim_{\mathbf{r}\to(8,8)} \frac{x^{1/3}-y^{1/3}}{x^{2/3}-y^{2/3}}$$

exists. If the limit exists, compute it. Otherwise, prove that the limit does not exist.

Solution

In some cases we can use a substitution to reduce a calculus-three limit to a calculus-one limit.

Example 2.2.12. Evaluate

$$\lim_{\mathbf{r}\to(0,0)} \frac{\sin\left(x^2y\right)}{x^2y}.$$

Solution

Proposition 2.2.13. *Suppose $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ is a function with domain D and $\mathbf{r}_0 \in \overline{D}$. If $\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r}) = L$, then for all paths $P \subset D$,*

$$\lim_{\mathbf{r}\to\mathbf{r}_0 : \mathbf{r}\in P} f(\mathbf{r}) = L.$$

Remark 2.2.14. If f is a function of a single variable, and x_0 is in the closure of f's domain, morally, there are at most two paths leading to x_0 from within f's domain; the path approaching x_0 from the left and the path approaching x_0 from the right. In this case Proposition 2.2.13 says that if $\lim_{x\to x_0} f(x) = L$ then both $\lim_{x\to x_0^-} f(x) = L$ and $\lim_{x\to x_0^+} f(x) = L$.

Remark 2.2.15. If we suspect that $\lim_{\mathbf{r}\to\mathbf{r}_0}$ does not exist, then Proposition 2.2.13 gives us two ways to prove this suspicion.

1. Find a path P in D leading to $\mathbf{r}_0$ along which the limit of $f(\mathbf{r})$ as $\mathbf{r} \to \mathbf{r}_0$ fails to exist.

2. Find two paths P_1 and P_2 along which the limits of $f(\mathbf{r})$ as $\mathbf{r} \to \mathbf{r}_0$ exist, but do not coincide.

Example 2.2.16. Show that

$$\lim_{\mathbf{r}\to\mathbf{0}} \frac{(x+y)^2}{x^2+y^2}$$

does not exist.

Solution

Example 2.2.17. Consider the function

$$f(x, y) = \frac{3xy^2}{x^2 + y^4}.$$

Determine whether $\lim_{\mathbf{r} \to \mathbf{0}} f(\mathbf{r})$ exists. If the limit exists, compute it. Otherwise, prove that the limit does not exist.

Solution

Proposition 2.2.18 (Algebraic Properties of Limits). *Suppose $L, M \in \mathbb{R}$, $\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r}) = L$ and $\lim_{\mathbf{r}\to\mathbf{r}_0} g(\mathbf{r}) = M$. Let c be a constant. The following properties hold*

1. $\lim_{\mathbf{r}\to\mathbf{r}_0} (f(\mathbf{r}) \pm g(\mathbf{r})) = L \pm M$.
2. $\lim_{\mathbf{r}\to\mathbf{r}_0} (f(\mathbf{r})g(\mathbf{r})) = LM$.
3. $\lim_{\mathbf{r}\to\mathbf{r}_0} cf(\mathbf{r}) = cL$.
4. $\lim_{\mathbf{r}\to\mathbf{r}_0} (f(\mathbf{r})/g(\mathbf{r})) = L/M$ *provided* $M \neq 0$.
5. $\lim_{\mathbf{r}\to\mathbf{r}_0} (f(\mathbf{r})^n) = L^n$ *for all integers* n.
6. $\lim_{\mathbf{r}\to\mathbf{r}_0} \left(f(\mathbf{r})^{n/m}\right) = L^{n/m}$ *provided m and n have no common factors and, if m is even, then L is not negative.*

Definition 2.2.19. let $D \subset \mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a function with domain D. We say f is *continuous at* $\mathbf{r}_0$ if

$$\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r}) = f(\mathbf{r}_0). \tag{2.1}$$

If f is continuous on its entire domain, we say f is *continuous*.

Remark 2.2.20. Equation (2.1) actually says three things.

1. $f(\mathbf{r}_0)$ exists (i.e. $\mathbf{r}_0$ is in f's domain).
2. $\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r})$ exists.
3. $\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r})$ and $f(\mathbf{r}_0)$ coincide.

If any one of these three conditions fails then f is not continuous at $\mathbf{r}_0$.

Example 2.2.21. Find all values of a such that

$$f(\mathbf{r}) = \begin{cases} \frac{xy - 4y^2}{\sqrt{x} - 2\sqrt{y}} & \mathbf{r} \neq \mathbf{0} \\ a & \mathbf{r} = \mathbf{0} \end{cases}$$

is continuous on $\mathbb{R}^2$.

Solution

Example 2.2.22. Find all values of a such that

$$f(\mathbf{r}) = \begin{cases} \frac{3-2(x^2+y^2)-3\cos(x^2+y^2)}{x^2+y^2} & \mathbf{r} \neq \mathbf{0} \\ a & \mathbf{r} = \mathbf{0} \end{cases}$$

is continuous on $\mathbb{R}^2$.

Solution

2.2.1 Exercises

1. Evaluate the following limits

   (a) $\lim_{\mathbf{r}\to(3,1)} \frac{x^2-7xy+12y^2}{3-y}$

   (b) $\lim_{\mathbf{r}\to(2,2)} \frac{y^2-4}{xy-2x}$

   (c) $\lim_{\mathbf{r}\to(4,5)} \frac{\sqrt{x+y}-3}{x+y-9}$

2. Show that the limits do not exist

   (a) $\lim_{\mathbf{r}\to\mathbf{0}} \frac{y^4-2x^2}{y^4+x^2}$.

   (b) $\lim_{\mathbf{r}\to\mathbf{0}} \frac{y^3+x^3}{xy^2}$

3. Determine whether the following exist. If the limits exist, find their values. Otherwise, prove that the limits do not exist.

   (a) $\lim_{\mathbf{r}\to\mathbf{0}} \frac{y}{\sqrt{x^2-y^2}}$

   (b) $\lim_{\mathbf{r}\to\mathbf{0}} \frac{x^3-y^3}{x^3+y^3}$

   (c) $\lim_{\mathbf{r}\to\mathbf{0}} \frac{1-\cos(x^2+y^2)}{x^2+y^2}$

4. Find a such that

$$f(x,y) = \begin{cases} \frac{\tan(xy)}{xy} & xy \neq 0 \\ a & xy = 0 \end{cases}$$

   is continuous. If no such a exists, explain why.

5. Find a such that

$$f(\mathbf{r}) = \begin{cases} \frac{3xy^2}{x^2+y^4} & \mathbf{r} \neq \mathbf{0} \\ a & \mathbf{r} = \mathbf{0} \end{cases}$$

   is continuous. If no such a exists, explain why.

6. Find a such that

$$f(x,y) = \begin{cases} e^{-\frac{1}{\|\mathbf{r}\|^2}} & x^2+y^2 \neq 0 \\ a & x^2+y^2 = 0 \end{cases}$$

   is continuous. If no such a exists, explain why.

2.3 Partial Derivatives

Recall the limit definition the derivative of a single-variable function at a point a.

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(x)}{x - a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

If the limit exists, $f'(a)$ gives the slope of the line tangent to the graph of f at $(a, f(a))$.

Definition 2.3.1. Suppose $D \subset \mathbb{R}^2$ and $\mathbf{r}_0 = (x_0, y_0)$ is an interior point of D. The *x-partial derivative* of f at (x_0, y_0), denoted either $f_x(x_0, y_0)$ or $\frac{\partial f}{\partial x}(x_0, y_0)$ is

$$f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists. Similarly, the *y-partial derivative* of f at (x_0, y_0) is

$$f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists. When they exist, $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are called the *first-order* partial derivatives of f at (x_0, y_0).

Remark 2.3.2.

1. When it exists, $\frac{\partial f}{\partial x}(x_0, y_0)$ is the slope of the line that is both parallel to the xz-plane and tangent to the graph of f at the point $(x_0, y_0, f(x_0, y_0))$. Moreover, the equation of this line is

$$\mathbf{r}(t) =$$

2. Similarly, when it exists, $\frac{\partial f}{\partial y}(x_0, y_0)$ is the slope of the line that is both parallel to the yz-plane and tangent to the graph of f at the point $(x_0, y_0, f(x_0, y_0))$. Moreover, the equation of this line is

$$\mathbf{r}(t) =$$

Remark 2.3.3. If f is a function of n variables $x_1, \cdots, x_n$ and if $\mathbf{r}_0 = (x_1^0, x_2^0, \cdots, x_n^0)$ is an interior point of f's domain, then there are n possible first-order partial derivatives at $\mathbf{r}_0$. They are given by

$$\frac{\partial f}{\partial x_j}(\mathbf{r}_0) = \lim_{h \to 0} \frac{f(\mathbf{r}_0 + h\mathbf{e}_j) - f(\mathbf{r}_0)}{h} \qquad j = 1, 2, \cdots, n,$$

provided the limits exist.

Example 2.3.4. Use the limit-definition of partial derivative to compute the first-order partial derivatives of

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

at $(0,0)$.

Solution

In the simplest cases we don't need to use the limit-definition of partial derivative to compute partial derivatives. In these cases, computing a partial derivative of a function with respect to a certain variable simply amounts to treating all other variables as constants and differentiating (using the usual calculus 1 rules) with respect to the desired variable.

Example 2.3.5. Compute both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for

$$f(x, y) = y \cos\left(\frac{x}{y^2}\right).$$

Solution

Example 2.3.6. Compute the first-order partial derivates of $f(x, y) = \|\mathbf{r}\|^3$.

Solution

For "nice enough" functions f, the partial derivatives of f_x and f_y exist and themselves have partial derivatives. In this case we consider the *second-order* partial derivatives of f as defined in 2.3.1. For example, the y-partial of f_x at (x_0, y_0) is denoted either $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$ or $f_{xy}(x_0, y_0)$ and is given by

$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x_0, y_0) = \lim_{h \to 0} \frac{f_x(x_0, y_0 + h) - f_x(x_0, y_0)}{h},$$

provided the limit exists. If f is a function of two variables x and y, there are four possible second-order partial derivatives for f. They are

$$\frac{\partial^2 f}{\partial x^2}, \quad \underbrace{\frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x},}_{\text{'mixed' partials}} \quad \frac{\partial^2 f}{\partial y^2}.$$

If f is a function of n variables $x_1, x_2, \cdots, x_n$, there are n^2 possible second-order partials of f. They are

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \qquad i = 1, 2, \cdots, n; \ j = 1, 2, \cdots n.$$

Example 2.3.7. Compute all first-order and all second-order partial derivatives of

$$f(x, y) = 3x^3 - 4xy^2 + 2y^3.$$

Solution

Remark 2.3.8. Third, fourth, and even higher-order partial derivatives are defined in the obvious way.

Theorem 2.3.9 (Clairaut's Theorem). *Suppose f is a function whose domain is a subset of $\mathbb{R}^2$. If the second-order partial derivatives exist and are continuous at the point $\mathbf{r}_0$ then the second-order mixed-partials are equal at $\mathbf{r}_0$. That is,*

$$\frac{\partial^2 f}{\partial x \partial y}(\mathbf{r}_0) = \frac{\partial^2 f}{\partial y \partial x}(\mathbf{r}_0).$$

Recovering a Function from its Partial Derivatives

We wish to answer the following question. Given n functions, $f_1(\mathbf{r}), f_2(\mathbf{r}), \cdots, f_n(\mathbf{r})$, is there a function $F(\mathbf{r})$ of n variables whose first-order partial derivatives are the given functions? Of course, if such an F exists, we will want to find it.
If $n = 1$, then this is answered by antidifferentiation as in calculus one.

Example 2.3.10. Given $f(x) = \frac{1}{1+x^2}$, find a function $F(x)$ whose derivative is $f(x)$.

Solution

If $n > 1$, the problem becomes a little more subtle. In this case (assuming the prescribed functions $f_j(\mathbf{r})$ have continuous first-order partials), the following integrability condition (which follows from Clairaut's Theorem) must be met.

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \qquad \text{for all } \; i = 1, 2, \cdots, n; \; j = 1, 2, \cdots, n.$$

Example 2.3.11. Let $f_1(x, y) = 2x + y$ and $f_2(x, y) = 2y - x$. Determine whether there is a function $F(x, y)$ such that both $F_x(x, y) = f_1(x, y)$ and $F_y(x, y) = f_2(x, y)$. If such an F exists, find it. Otherwise show that no such F exists.

Solution

Example 2.3.12. Let $\mathbf{r} = (x, y, z)$ and consider the functions

$$f_1(\mathbf{r}) = 6xy, \quad f_2(\mathbf{r}) = 3x^2 + z^3, \quad f_3(\mathbf{r}) = 3yz^2 + 1.$$

Find a function $F(\mathbf{r})$ such that

$$\frac{\partial F}{\partial x}(\mathbf{r}) = f_1(\mathbf{r}), \quad \frac{\partial F}{\partial y}(\mathbf{r}) = f_2(\mathbf{r}), \quad \text{and} \quad \frac{\partial F}{\partial z}(\mathbf{r}) = f_3(\mathbf{r}).$$

If no such function exists, explain why.

Solution

2.3.1 Exercises

1. Compute all second-order partial derivatives of the given function.

   (a) $f(x, y) = \arctan(xy)$.

   (b) $f(x, y) = x^y$.

   (c) $f(x, y, z) = (x + y)/(x + 2z)$.

2. Let $f_1(x, y, z) = yz + 2x$, $f_2(x, y, z) = xz + 3y^2$ and $f_3(x, y, z) = xy + 4z^3$. Determine whether there is a function $F(x, y, z)$ such that $F_x = f_1$, $F_y = f_2$ and $F_z = f_3$. If F exists, find it. Otherwise, prove that F does not exist.

3. Let $f(x, y) = \ln(x^2 + y^2)$ for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Compute
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

4. Let
$$\phi(t, x) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{2t}} \qquad \text{for } t > 0,\ -\infty < x < \infty.$$
Compute the ratio ϕ_t/ϕ_{xx} for pairs (t, x) satisfying $\phi_{xx}(t, x) \neq 0$.

5. Let
$$\Psi(x, y, t) = \frac{1}{4\pi t} e^{-\frac{(x^2+y^2)}{4t}} \qquad \text{for } (x, y) \in \mathbb{R}^2,\ t > 0.$$

   (a) Compute $\frac{\partial \Psi}{\partial t}$.

   (b) Compute $\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}$.

6. Let
$$f(x, y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

   (a) Use "symbol-manipulation" to compute $f_x(x, y)$ and $f_y(x, y)$ for $(x, y) \neq 0$. What happens if you try to plug $(0, 0)$ into your formulas for $f_x(x, y)$ and $f_y(x, y)$?

   (b) Use the limit-definition of partial derivative to compute $f_x(0, 0)$ and $f_y(0, 0)$.

   (c) Use the limit definition of partial derivative (applied to the appropriate first-order partials of f) to compute both $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$. Compare your answer to the conclusion of Clairaut's Theorem (theorem 2.3.9). Is this a contradiction?

2.4 Linearization of Multivariate Functions

Example 2.4.1. Let $f(x) = \ln(x^2)$ for $|x| > 0$. Find the equation of the line tangent to the graph of f at $x_0 = 4$.

Solution

Routine computations give both

$$f(4) = \ln(16) \quad \text{and} \quad f'(4) = \frac{1}{2}.$$

The equation of the line tangent to the graph of f at $x_0 = 4$ is therefore

$$\begin{aligned} L_4(x) &= f(4) + f'(4)(x-4) \\ &= \ln 16 + \frac{1}{2}(x-4). \end{aligned}$$

Remark 2.4.2. The line $L_4(x) = \ln(16) + \frac{1}{2}(x-4)$ is called the linearization of f at $x_0 = 4$. We want to develop a similar concept in the case that f is a function of more than one variable.

Example 2.4.3. Let $D \subset \mathbb{R}^2$ be an open set and let $f : D \to \mathbb{R}$. Let $(x_0, y_0) \in D$ and assume that both $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist. Derive an equation for the plane tangent to the graph of f at the point $(x_0, y_0, f(x_0, y_0))$.

Solution

Definition 2.4.4. Let $D \subset \mathbb{R}^2$ be an open set and let $\mathbf{r}_0 = (x_0, y_0) \in D$. Let $f : D \to \mathbb{R}$ be a function whose first-order partials exist at $\mathbf{r}_0$. The *linearizaton* of f at $\mathbf{r}_0$ is

$$\begin{aligned} L_{\mathbf{r}_0}(\mathbf{r}) &= f(\mathbf{r}_0) + \left\langle \frac{\partial f}{\partial x}(\mathbf{r}_0), \frac{\partial f}{\partial y}(\mathbf{r}_0) \right\rangle \cdot (\mathbf{r} - \mathbf{r}_0). \\ &= f(x_0, y_0) + \left\langle \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right\rangle \cdot \langle x - x_0, y - y_0 \rangle . \end{aligned}$$

Remark 2.4.5. $L_{\mathbf{r}_0}(\mathbf{r})$ is the function whose graph is the plane tangent to the graph of f at the point $(x_0, y_0, f(x_0, y_0))$.

Example 2.4.6. Find the linearization of $f(x, y)$ at the point $(0, 0)$ if

$$f(x, y) = \frac{2y + 3}{4x + 1}.$$

Solution

Linearizations can be used to easily estimate quantities that are tedious to compute.

Example 2.4.7. Use an appropriate linearization to estimate the value of

$$\sqrt{(2.03)^2 + (1.98)^2}.$$

Solution

Remark 2.4.8. For a function f of a single variable (i.e. a calculus 1 function), existence of $f'(x_0)$ is enough to guarantee that f is differentiable at x_0. The situation is more complicated for functions of more than one variable.

Definition 2.4.9. Let $D \subset \mathbb{R}^2$ be an open set and let $\mathbf{r}_0 = (x_0, y_0) \in D$. Suppose $f : D \to \mathbb{R}$ has a linearization $L_{\mathbf{r}_0}(\mathbf{r})$ at $\mathbf{r}_0$. We say f is *differentiable* at $\mathbf{r}_0$ if

$$\lim_{\mathbf{r}\to\mathbf{r}_0} \frac{f(\mathbf{r}) - L_{\mathbf{r}_0}(\mathbf{r})}{\|\mathbf{r} - \mathbf{r}_0\|} = 0. \tag{2.2}$$

Remark 2.4.10.

1. Equation (2.2) says that as $\mathbf{r} \to \mathbf{r_0}$, the graph of f approaches the graph of the plane tangent to f at $(x_0, y_0, f(x_0, y_0))$ faster than $\mathbf{r}$ approaches $\mathbf{r_0}$.

2. To show a function f is differentiable at $\mathbf{r}_0$,

   (a) Compute $L_{\mathbf{r}_0}(\mathbf{r})$, the linearization of f at $\mathbf{r}_0$ (if the linearization does not exist, then f is not differentiable).

   (b) Show that the limit in equation (2.2) holds (if the limit does not hold, then f is not differentiable).

Example 2.4.11. Show that

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is differentiable at $(0,0)$.

Solution

Proposition 2.4.12. *If f is differentiable at $\mathbf{r}_0$, then f is continuous at $\mathbf{r}_0$.*

Remark 2.4.13. The converse of proposition 2.4.12 is false; some functions are continuous but not differentiable. In fact as the following example shows, some functions are continuous and have first-order partials, yet still fail to be differentiable.

Example 2.4.14. Let

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that f is continuous at $(0,0)$ and has first-order partial derivatives at $(0,0)$, but that f is not differentiable at $(0,0)$.

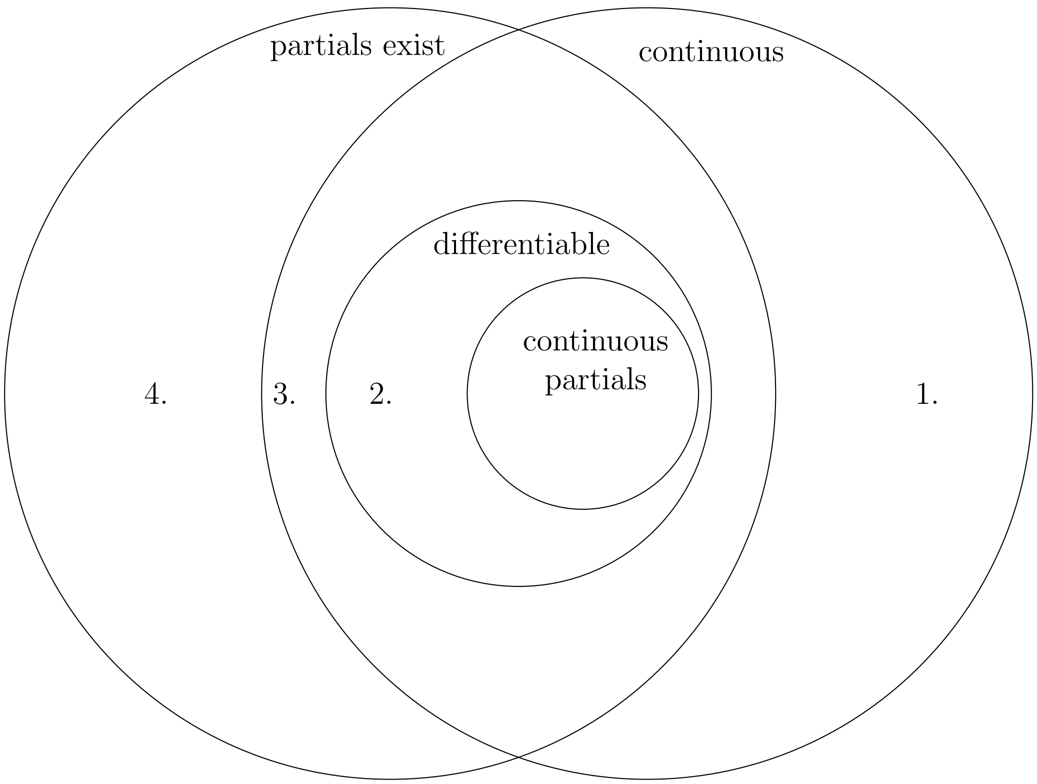
Solution

Proposition 2.4.15. *Let $D \subset \mathbb{R}^n$ be an open set and let $f : D \to \mathbb{R}$. If the first order partials of f exist and are continuous in D, then f is differentiable in D.*

Henceforth you may use the following facts without proof (of course, if you are specifically asked to prove one of these facts, you must do so).

1. Polynomials have partial derivatives of all orders and are therefore differentiable.
2. Finite sums, differences and products of differentiable functions are differentiable.
3. The quotient of differentiable functions is differentiable on its domain.
4. The composition of differentiable functions is differentiable on its domain.
5. Trigonometric functions logarithmic functions, exponential functions and power functions are differentiable on their domains.

Venn Diagram of Regularity



1. $f(x,y) = |x|$

2. $f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

3. $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

4. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

2.4.1 Exercises

1. For each given f, find an equation of the plane tangent to the graph of f at the given point. [*Hint: consider remark 2.4.5.*]

   (a) $f(x,y) = 2 + 2x^2 + \frac{1}{2}y^2$ at $(-\frac{1}{2}, 1, 3)$.

   (b) $f(x,y) = e^{xy}$ at $(0,1,1)$.

   (c) $f(x,y) = \ln(1+xy)$ at $(1,2,\ln 3)$.

   (d) $f(x,y) = \arctan(xy)$ at $(1,1,\pi/4)$.

2. Use an appropriate linearization to estimate the following quantities

   (a) $-4(1.05)^2 - 8(3.95)^2$.

   (b) $\sqrt{(3.06)^2 + (3.92)^2}$.

3. Show that
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$
is not continuous at $(0,0)$, but has first-order partials at $(0,0)$.

4. Let
$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$
In example 2.4.11, we showed that f is differentiable at $(0,0)$. Show that f does not have continuous first-order partials at $(0,0)$ (so differentiability is insufficient to ensure continuity of first-order partials). You may use without proof the fact that
$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2} \cos\left(\frac{1}{x^2+y^2}\right)$$
does not exist.

2.5 Chain Rule and Implicit Function Theorem

Example 2.5.1. Let $f(x, y) = x^3 + xy^2$ and suppose x and y are both functions of r and θ and are given by $x = r\cos\theta$, $y = r\sin\theta$. Let

$$F(r, \theta) = f(x(r, \theta), y(r, \theta)).$$

Compute both $\frac{\partial F}{\partial r}$ and $\frac{\partial F}{\partial \theta}$.

Solution

Example 2.5.2. As in example 2.5.1, let $f(x,y) = x^3 + xy^2$ and suppose x and y are both functions of r and θ and are given by $x = r\cos\theta$, $y = r\sin\theta$. Let

$$F(r,\theta) = f(x(r,\theta), y(r,\theta)).$$

Compute both

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}.$$

Compare your answer to the answer from example 2.5.1.

Solution

Theorem 2.5.3 (Two-Variable Chain Rule). *Suppose f is a function of x and y and that each of x and y are functions of u and v. Then*

$$F(u,v) = f(x(u,v), y(u,v))$$

defines a function of u and v. If f is differentiable at (x,y) and both of $x(u,v)$ and $y(u,v)$ are differentiable functions of u and v, then $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ exist and are given by

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

and

$$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

Theorem 2.5.4 (Chain Rule). *Suppose f is a function of the n variables $x_1, \cdots, x_n$ and that for each $j = 1, \cdots, n$, x_j is a function of the m variables $u_1, \cdots u_m$. Then*

$$F(u_1, \cdots, u_m) = f(x_1(u_1, \cdots, u_m), \cdots, x_n(u_1, \cdots, u_m))$$

defines a function of $u_1, \cdots u_m$. If f is differentiable and if each $x_j(u_1, \cdots, u_m)$ is differentiable, then for each $k = 1, \cdots, m$, $\frac{\partial F}{\partial u_k}$ exists and is given by

$$\frac{\partial F}{\partial u_k}(u_1, \cdots, u_m) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial u_k}.$$

Example 2.5.5. Suppose $f(x, y, z)$ is differentiable and that

$$f_x((1, 2, 3)) = 1, \quad f_y((1, 2, 3)) = 2, \quad \text{and} \quad f_z((1, 2, 3)) = -2.$$

Suppose x, y, z are functions of s and t given by

$$x(s, t) = t^2 s, \quad y(s, t) = s + t, \quad \text{and} \quad z(s, t) = 3s.$$

Compute both the rate of change of f with respect to s and the rate of change of f with respect to t at the point $(1, 2, 3)$.

Solution

Example 2.5.6. Let
$$f(x, y, z) = \frac{z^2}{(1 + x^2 + 2y^2)}.$$
Find the rate of change of f along the curve $\mathbf{r}(t) = (\sin t, \cos t, e^t)$ in the direction of increasing t.

Solution

2.5.1 Implicit Differentiation

Lets warm up with a calculus one problem.

Example 2.5.7. Consider the circle $x^2 + y^2 = 1$ in two-dimensional space. Find the slope of the line tangent to the circle at the point $(\sqrt{3}/2, 1/2)$. What happens if we try to do this at the point $(1, 0)$? Does the given equation implicitly define y as a function of x near the point $(1, 0)$?

Solution

Theorem 2.5.8 (Implicit Function Theorem). *Let* $\mathbf{r} = (x_1, \cdots, x_n)$ *and suppose* $F(\mathbf{r}, z)$ *is a function of* $n+1$ *variables with continuous first-order partial derivatives. Suppose further that there is a point* $(\mathbf{r}_0, z_0)$ *such that* $F(\mathbf{r}_0, z_0) = 0$*. If* $\frac{\partial F}{\partial z}(\mathbf{r}_0, z_0) \neq 0$*, then there exists a differentiable function* $g(\mathbf{r})$ *(independent of* z*) such that* $g(\mathbf{r}_0) = z$ *and, for all* $\mathbf{r}$ *near* $\mathbf{r}_0$*,*

$$F(\mathbf{r}, z) = 0 \ \text{ if and only if } \ z = g(\mathbf{r}).$$

Moreover, for $\mathbf{r}$ *near* $\mathbf{r}_0$*, the partial derivatives of* z *are given by*

$$\frac{\partial z}{\partial x_j} = -\frac{\frac{\partial F}{\partial x_j}}{\frac{\partial F}{\partial z}}.$$

Example 2.5.9. Show that the equation $x^2 + y^2 + z^2 = 9$ implicitly defines z as a function of x and y near the point $(1, 2, 2)$ and find the linearization of z at $(1, 2, 2)$.

Solution

Example 2.5.10. Show that the equation

$$z(3x - y) = \pi \sin(xyz)$$

implicitly defines z as a differentiable function of x and y near the point $(1, 1, \pi/2)$ and find an equation of the plane tangent to the graph of z at $(1, 1, \pi/2)$.

Solution

2.5.2 Exercises

1. Suppse $f(x,y) = e^{-(x^2+y^2)}$ and $x = t$, $y = \sqrt{t}$. Compute $\frac{\partial f}{\partial t}$.

2. Suppose $f(x,y) = \ln(x+y+z)$ and that $x = \cos^2 t$, $y = \sin^2 t$ and $z = t^2$. Compute $\frac{\partial f}{\partial t}$.

3. Suppose $f(x,y,z) = e^{x+y+z}$ and that $x = vw$, $y = uw$ and $z = uv$. Compute $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ and $\frac{\partial f}{\partial w}$.

4. Let $f(x,y,z) = xy+xz+yz$ and suppose $x = s^2-t^2$, $y = s^2+t^2$ and $z = s^2t^2$. Compute $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

5. Find an equation of the plane tangent to the surface

$$z^3 + (x+y)z^2 + x^2 + y^2 = 13$$

at the point $(2,2,1)$.

6. Suppose z is implicitly defined as a function of x and y by the given equations. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

   (a) $x^3 + y^3 + z^3 = xyz$.

   (b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

   (c) $xyz = \sin(x+y+z)$.

7. Show that the equation

$$z^3 - xz + y = 0$$

implicitly defines z as a function of x and y near the point $(3,-2,2)$. Find the linearzation of z at $(3,-2,2)$ and use it to approximate $z(2.8,-2.3)$.

2.6 Directional Derivatives and Gradients

Definition 2.6.1. Let f be a function whose domain is an open set $D \subset \mathbb{R}^n$ and suppose $\mathbf{r}_0 \in D$. The *directional derivative of f at $\mathbf{r}_0$ in the direction of the unit vector $\widehat{\mathbf{u}}$* is

$$D_{\mathbf{u}} f(\mathbf{r}_0) = \lim_{h\to 0} \frac{1}{h}\left(f(\mathbf{r}_0 + h\widehat{\mathbf{u}}) - f(\mathbf{r}_0)\right),$$

provided the limit exists.

Remark 2.6.2. If $\widehat{\mathbf{u}} = \widehat{\mathbf{i}} = \langle 1,0,0\rangle$, then the formula in definition 2.6.1 gives the usual partial derivative of f with respect to x.

Proposition 2.6.3. *Let f be as in Definition 2.6.1. If f is differentiable at $\mathbf{r}_0$ then*

$$\begin{aligned} D_{\mathbf{u}}f(\mathbf{r}_0) &= \frac{\partial f}{\partial x_1}(\mathbf{r}_0)u_1 + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{r}_0)u_n \\ &= \left\langle \frac{\partial f}{\partial x_1}(\mathbf{r}_0), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{r}_0) \right\rangle \cdot \langle u_1, \cdots, u_n \rangle . \end{aligned}$$

Proof. Let $\mathbf{r}(h) = \mathbf{r}_0 + h\widehat{\mathbf{u}}$. By definition 2.6.1,

$$\begin{aligned} D_{\mathbf{u}}f(\mathbf{r}_0) &= \lim_{h\to 0} \frac{1}{h} \left(f(\mathbf{r}(h)) - f(\mathbf{r}(0)) \right) \\ &= \frac{d}{dh} \left. f(\mathbf{r}(h)) \right|_{h=0} . \end{aligned}$$

Using the chain rule (theorem 2.5.4) we get

$$\begin{aligned} \frac{d}{dh} f(\mathbf{r}(h)) &= \frac{\partial f}{\partial x_1}(\mathbf{r}(h))\frac{\partial r_1}{\partial h} + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{r}(h))\frac{\partial r_n}{\partial h} \\ &= \frac{\partial f}{\partial x_1}(\mathbf{r}(h))u_1 + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{r}(h))u_n. \end{aligned}$$

Evaluating this formula at $h = 0$ and using $\mathbf{r}(0) = \mathbf{r}_0$ we have

$$D_{\mathbf{u}}f(\mathbf{r}_0) = \frac{\partial f}{\partial x_1}(\mathbf{r}_0)u_1 + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{r}_0)u_n.$$

∎

Example 2.6.4. Let $f(x, y) = 25 - x^2 - y^2$. Compute the directional derivative of f at the point $(1, 1)$ in the direction of $\mathbf{u} = \langle -1, -1 \rangle$ in two ways; using Definition 2.6.1 and using Proposition 2.6.3.

Solution

Example 2.6.5. Compute the directional derivative of

$$f(x, y, z) = x^2 + 3xz + z^2 y$$

at the point $(1, 1, -1)$ in the direction toward the point $(3, -1, 0)$. Does f increase or decrease in this direction?

Solution

Definition 2.6.6. Let $D \subset \mathbb{R}^n$ be an open set and let $\mathbf{r}_0 \in D$. If the first order partial derivatives of f exist at $\mathbf{r}_0$, the *gradient of f at* $\mathbf{r}_0$ is denoted $\nabla f(\mathbf{r}_0)$ and is given by

$$\nabla f(\mathbf{r}_0) = \left\langle \frac{\partial f}{\partial x_1}(\mathbf{r}_0), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{r}_0) \right\rangle.$$

Remark 2.6.7. According to Proposition 2.6.3, if f is differentiable then the directional derivative of f in the direction of $\mathbf{u}$ can be computed via performing the dot-product of ∇f with $\widehat{\mathbf{u}}$

$$D_{\mathbf{u}} f(\mathbf{r}_0) = \nabla f(\mathbf{r}_0) \cdot \widehat{\mathbf{u}}.$$

Moreover, as in Definition 2.4.4, the linearization of f at $\mathbf{r}_0$ may be written as

$$L_{\mathbf{r}_0}(\mathbf{r}) = f(\mathbf{r}_0) + \nabla f(\mathbf{r}_0) \cdot (\mathbf{r} - \mathbf{r}_0).$$

Example 2.6.8. Let $z = f(x, y) = 25 - x^2 - y^2$.

1. Compute $\nabla f(x, y)$.
2. Find a parameterization $\mathbf{r}(t)$ for the level curve of f corresponding to level $z = 9$. Plot this level curve in xy-space.
3. On the same plot from the previous part, plot a unit vector whose direction coincides with the direction of $\nabla f(2\sqrt{2}, -2\sqrt{2})$. Plot this vector with initial point $(2\sqrt{2}, -2\sqrt{2})$.
4. Compute $\nabla f \cdot \mathbf{r}'$ at $(2\sqrt{2}, -2\sqrt{2})$ and compare the result to your plot.

Solution

In fact, the phenomenon observed in example 2.6.8 occurs frequently as the next theorem states.

Theorem 2.6.9. *Suppose f is a function whose domain is an open set $D \subset \mathbb{R}^n$ and suppose $\mathbf{r}_0 \in D$. If f is differentiable at $\mathbf{r}_0$ and $\nabla f(\mathbf{r}_0) \neq \mathbf{0}$, then*

1. *The maximal rate of change of f at $\mathbf{r}_0$ is attained in the direction of $\nabla f(\mathbf{r}_0)$. The value of this maximal rate is $\|\nabla f(\mathbf{r}_0)\|$.*
2. *The minimal rate of change of f at $\mathbf{r}_0$ is attained in the direction of $-\nabla f(\mathbf{r}_0)$. The value of this minimal rate is $-\|\nabla f(\mathbf{r}_0)\|$.*
3. *If there is $\delta > 0$ such that f has continuous first-order partial derivatives in $B_\delta(\mathbf{r}_0)$ then $\nabla f(\mathbf{r}_0)$ is orthogonal to the (tangent line/plane to) the level set of f at $\mathbf{r}_0$.*

Example 2.6.10. Find the maximal and minimal rates of change and the direction in which these rates occur for $f(x, y, z) = x\sin(yz)$ at the point $(1, 2, \frac{\pi}{3})$.

Solution

Example 2.6.11. Find an equation of the tangent plane to the surface $x^2 + y^2 + z^2 = 169$ at the point $(3, 4, 12)$. *Hint: view the given surface as a level set for an appropriate function and use Theorem 2.6.9.*

Solution

Remark 2.6.12. The existence of all directional derivatives is insufficient to deduce differentiability as the following example shows.

Example 2.6.13. Let

$$f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that f has directional derivatives in all directions at $(0,0)$ but that f is not differentiable at $(0,0)$.
Note: Proposition 2.6.3 does not apply in this situation since we don't know f is differentiable.

Solution

2.6.1 Exercises

1. Let $f(x, y) = x/(1 + xy)$. Compute $\nabla f(1, 1)$ and the directional derivative of f at $(1, 1)$ in the direction toward the point $(2, 1)$. Does f increase or decrease in this direction?

2. Let $f(x, y, z) = \arctan(1 + x + y^2 + z^2)$. Compute $\nabla f(1, -1, 1)$ and the directional derivative of f at $(1, -1, 1)$ in the direction toward the point $(1, 1, 1)$. Does f increase or decrease in this direction?

3. Find the maximum and minimum rates of change of f at the given point and the direction in which these rates occur. Find the directions in which the rate of change of f is zero.

   (a) $f(x, y) = x/y^2$ at $(2, 1)$.

   (b) $f(x, y) = x^y$ at $(2, 1)$.

   (c) $f(x, y) = xz/(1 + yz)$ at $(1, 2, 3)$.

4. Let
$$f(x, y, z) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}.$$
Find the rate of change of f in the direction of the vector which is tangent to the curve $\mathbf{r}(t) = \langle t, 2t^2, -2t^2 \rangle$ at the point $(1, 2, -2)$.

5. Let $f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$. Find all points at which ∇f is orthogonal to the z-axis.

6. Find an equation of the plane tangent to the surface $x^2 - 2y^2 + z^2 + yz = 2$ at the point $(2, 1, -1)$. *Hint: View the surface as a level surface for the function* $w = f(x, y, z) = x^2 - 2y^2 + z^2 + yz$ *and consider theorem 2.6.9.*

2.7 Optimization

Theorem 2.7.1 (Extreme Value Theorem). *If D is a bounded subset of $\mathbb{R}^2$ and if f is continuous on $\overline{D}$, the closure of D, then there are points x_* and x^* in $\overline{D}$ such that*

$$\begin{aligned} f(x_*) &= \min_{x \in \overline{D}} f(x) \\ f(x^*) &= \max_{x \in \overline{D}} f(x). \end{aligned}$$

After completing Subsections 2.7.1 and 2.7.3, we will be able to find such x_* and x^* for some simple instances of f and D.

2.7.1 Local Optimization

Definition 2.7.2. Suppose f is defined on an open set $D \subset \mathbb{R}^2$ and $(x_0, y_0) \in D$.

1. We call (x_0, y_0) a *local minimizer* of f if there is $\delta > 0$ (small) such that if $(x, y) \in B_\delta(x_0, y_0)$, then $f(x, y) \geq f(x_0, y_0)$.
2. We call (x_0, y_0) a *local maximizer* of f if there is $\delta > 0$ (small) such that if $(x, y) \in B_\delta(x_0, y_0)$, then $f(x, y) \leq f(x_0, y_0)$.
3. (x_0, y_0) is called a *local optimizer* or a *local extremizer* of f if it is either a local maximizer or a local minimizer of f.

Definition 2.7.3. Suppose f is defined on an open set $D \subset \mathbb{R}^2$ and $(x_0, y_0) \in D$. We say (x_0, y_0) is a *critical point* of f if either

$$\nabla f(x_0, y_0) = \mathbf{0} \qquad \text{or} \qquad \nabla f(x_0, y_0) \text{ DNE.}$$

Theorem 2.7.4. *Suppose f is defined on an open set $D \subset \mathbb{R}^2$. If $(x_0, y_0) \in D$ is a local extremizer of f then (x_0, y_0) is a critical point of f.*

Remark 2.7.5. Theorem 2.7.4 says that when attempting to find local optimizers for a given function f, we should start by looking for critical points of f. In general, not all critical points will be extremizers; the critical points of f are only candidates for local optimizers.

Example 2.7.6. Let

$$f(x, y) = e^{-(x-1)^2-(y+3)^2}.$$

Find all critical points of f and determine (by inspecting f) whether each is a local minimizer or a local minimizer.

Solution

Example 2.7.7. Let

$$f(x,y) = (x^2 + y^2)^{1/3} - 2.$$

Find all critical points of f and determine (by inspecting f) whether each is a local minimizer or a local minimizer.

Solution

Theorem 2.7.8 (Second Derivative Test). *Suppose f is defined and has continuous second-order partials in an open set $D \subset \mathbb{R}^2$. Suppose $(x_0, y_0) \in D$ and that $\nabla f(x_0, y_0) = \mathbf{0}$. Set*

$$\begin{aligned} \delta &= \det \begin{pmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{pmatrix} \\ &= f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2. \end{aligned}$$

1. *If $\delta > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimizer of f.*
2. *If $\delta > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximizer of f.*
3. *If $\delta < 0$ then $(x_0, y_0, f(x_0, y_0))$ is a saddle point.*
4. *If $\delta = 0$, then no conclusions can be made.*

Example 2.7.9. Find and classify all local extrema for $f(x, y) = \frac{1}{3}x^3 + xy^2 - x^2 - y^2$.

Solution

Example 2.7.10. Consider all rectangular boxes in the first octant of the xyz-space that have one vertex at the origin and the opposite vertex on the plane

$$6x + 8y + 3z = 24.$$

Find the maximal volume of such a box.

Solution

2.7.2 Exercises

1. Find and classify all local extremum.

   (a) $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 4$.

   (b) $f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 8$.

   (c) $f(x, y) = 2 - (x^2 + y^2)^{1/3}$.

2. Find all critical points. Use the second-derivative test (if applicable) to classify each as a local minimizer or a local maximizer.

   (a) $f(x, y) = (x + 1)^2 + (y - 4)^2$.

   (b) $f(x, y) = x^{2/3} + y^{2/3} - 1$.

   (c) $f(x, y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 10$.

   (d) $f(x, y) = (x + y)^{2/3} + 4$.

   (e) $f(x, y) = \arctan(xy)$.

   (f) $f(x, y) = xye^{-x^2-y^2}$.

   (g) $f(x, y) = y(1 + x^2 + y^2)^{-1}$.

   (h) $f(x, y) = (y - 1)(x^2 + y^2)^{-1}$.

   (i) $f(x, y) = xe^y - e^x$.

3. Find three positive numbers whose sum is 40 and whose product is maximal.

4. Find the dimensions of the rectangular box whose volume is maximal if the box lies in the first octant with one vertex at the origin and the other vertex on the plane $x + 2y + 7z = 14$. What is the volume of this box?

5. Let $\mathcal{P}$ be the plane

$$x - 3y - z = 1.$$

   and let $A = (1, 4, -7)$.

   (a) Use vector projections to compute the distance from A to $\mathcal{P}$.

   (b) Find a function d that depends only on x and y and whose value is the distance from A to a generic point on $\mathcal{P}$.

   (c) Minimize d to obtain both the distance from A to $\mathcal{P}$ and the point on $\mathcal{P}$ whose distance to A is the minimum possible.

2.7.3 Constrained Optimization

Example 2.7.11. Sketch the level curves of

$$f(x, y) = x^2 + y^2 - 2x + 2y + 5$$

corresponding to levels $z = 4, 7, 12, 19$. On the same axis, sketch the curve C given by $x^2 + y^2 = 4$. Of all the points on C, indicate which maximize f and which minimize f. How is C related to the level curves of f at these optimizing points?

Solution

Theorem 2.7.12. *Suppose f and g are n-variable functions that have continuous first-order partial derivatives. Suppose f has an extremum at $\mathbf{r}_0$ subject to the constraint $g(\mathbf{r}) = 0$. If $\nabla g(\mathbf{r}_0) \neq \mathbf{0}$, then there is $\lambda \in \mathbb{R}$ such that*

$$\nabla f(\mathbf{r}_0) = \lambda \nabla g(\mathbf{r}_0).$$

Method of Lagrange Multipliers. Let f and g be as in Theorem 2.7.12. When trying to optimize $f(\mathbf{r})$ subject to the constraint $g(\mathbf{r}) = 0$,

1. Solve the system of equations

$$\begin{cases} \nabla f(\mathbf{r}) = \lambda \nabla g(\mathbf{r}) \\ g(\mathbf{r}) = 0 \end{cases}$$

   for $\mathbf{r} = (x_1, \cdots, x_n)$ and λ. This will be a $(n+1) \times (n+1)$-system if f is an n-variable function.

2. For each n-tuple $\mathbf{r} = (x_1, \cdots, x_n)$ found in the first step, compute $f(\mathbf{r})$. The largest such value will correspond to a maximum of f (subject to $g = 0$) and the smallest such value will correspond to a minimum of f (subject to $g = 0$).

Example 2.7.13. Minimize $f(x, y) = -xy$ subject to the constraint $x^2 + y^2 = 4$.

Solution

Example 2.7.14. Minimize $f(x, y) = 2x + y$ subject to the constraint $xy = 32$.

Solution

Example 2.7.15. Use Lagrange multipliers to find the distance from the point $(1, 4)$ to the line $4x + y = 3$.

Solution

Example 2.7.16. Find the point on the sphere

$$x^2 + y^2 + z^2 = 9$$

whose distance to the point $(4, 0, -1)$ is maximal. What is the maximal distance?

Solution

Suppose f is a continuous function defined on a closed bounded subset D of $\mathbb{R}^n$. By the Extreme Value Theorem, f attains both a maximum value and a minimum value over D. To find these values and the points in D at which they occur, complete the following steps.

Step 1: Find all critical points of f which are in D.

Step 2: Use Lagrange multipliers to find the extrema of f on the boundary of D.

Step 3: For each n-tuple $(x_1, \cdots, x_n)$ found in the first two steps, evaluate $f(x_1, \cdots, x_n)$. The smallest of these values corresponds to an absolute minimum of f over D and the largest of these values corresponds to an absolute maximum of f over D.

Example 2.7.17. Consider the function

$$f(x, y) = x^2 + y^2 - 2x + 2y + 5$$

for $(x, y) \in D = \{(x, y) : x^2 + y^2 \leq 4\}$. Find the absolute minimum and the absolute maximum values and the points at which these values occur.

Solution

2.7.4 Exercises

1. Find the maximum and minimum values of f subject to the given constraint.
   (a) $f(x,y) = x^2y$ subject to $x^2 + y^2 = 4$.
   (b) $f(x,y) = x + y$ subject to $x^2 - xy + y^2 = 1$.
   (c) $f(x,y) = x^2 + y^2 - 1$ subject to $x^6 + y^6 = 1$.
   (d) $f(x,y) = 4xy$ subject to $x^2/9 + y^2/16 = 1$.
   (e) $f(x,y) = e^{xy}$ subject to $x^2 + y^2 = 8$.
   (f) $f(x,y) = x^2 + y^2 + z^2$ subject to $x + y + z = 6$.
   (g) $f(x,y) = x^2 + 10x + y^2 - 14y + 60$ subject to $x + y = 10$.

2. Find the absolute minimum and the absolute maximum values as well as the points at which these values occur for f over the set D.
   (a) $f(x,y) = -x^2 - y^2 + 2x - 2y - 2$; $D = \{(x,y) : x^2 + y^2 \leq 4\}$.
   (b) $f(x,y) = x^2 + 4y^2 + 3$; $D = \{(x,y) : x^2 + 4y^2 \leq 1\}$.
   (c) $f(x,y) = 2x^2 + y^2 + 2x - 3x$; $D = \{(x,y) : x^2 + y^2 \leq 1\}$.

3. Find the dimensions of the rectangular box of maximal volume that can be inscribed (with sides parallel to the coordinate axes) in the ellipsoid
$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1.$$

4. Use Lagrange multipliers to compute the distance from the point $(2,1,1)$ to the plane $x + y + z = 1$.

5. Use Lagrange multipliers to compute the distance from the point $(4,0,0)$ to the cone $z = \sqrt{x^2 + y^2}$.

2.8 Exam 2 Review

Problem 2.1. Find and graph the domain of the function
$$f(x,y,z) = \ln(z^2 - x^2 - y^2) + \sqrt{1 - x^2 - y^2 - z^2}.$$

Problem 2.2. Consider the function $f(x,y) = 2x^2 + 2y^2$. Graph both f and the level curves of f corresponding to the levels $z = 2, 4, 8$.

Problem 2.3. You must do this problem exactly the way I ask you to. Any other solutions will be given no credit. Let

$$f(x, y) = \sqrt{x^2 + y^2}\sin(x^2 + y^2).$$

1. Find a function g that depends only on $\|\mathbf{r}\| = \sqrt{x^2 + y^2}$ such that both $|f(x, y)| \leq g(\|\mathbf{r}\|)$ and such that $\lim_{\mathbf{r}\to 0} g(\|\mathbf{r}\|) = 0$.

2. Use the function g you found in the previous part to compute $\lim_{\mathbf{r}\to\mathbf{0}} f(x, y)$.

Problem 2.4. If it exists, find the value of a such that the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ a & \text{if } (x, y) = (0, 0) \end{cases}$$

is continuous at $(0, 0)$. If no such a exists explain why.

Problem 2.5. 1. Prove that the function

$$f(x, y) = xy\sqrt{x^2 + y^2}$$

is differentiable at $(0, 0)$.

2. Can we deduce based only on the result of part (a) that the first order partials of the given function are continuous?

Problem 2.6. 1. Find an equation of the plane $\mathcal{P}$ tangent to the surface $x^2 + y^2 - z^2 = 4$ at the point $(2, 1, 1)$.

2. How does the gradient vector of the function $f(x, y, z) = x^2 + y^2 - z^2$ at the point $(2, 1, 1)$ relate to the level surface for f at that point?

Problem 2.7. The point $(2, 0)$ satisfies the equation

$$2xy^2 - y^3 - x^2 + 4 = 0. \tag{2.3}$$

Determine, with justification, whether equation (2.3) defines y as a differentiable function of x near the point $x = 2$. If so, find $y'(2)$.

Problem 2.8. Find the unit vector $\widehat{\mathbf{u}}$ in the direction of the steepest descent of the function $f(x, y) = x^3 + y^5$ at the point $(2, 1)$.

Problem 2.9. Find all critical points of the function

$$f(x, y) = x^2 + y^2 - xy + 3x.$$

Use an appropriate test to determine whether each critical point is a local minimum, a local maximum or a saddle. Give the local minimum and local maximum values, if any, for f.

Problem 2.10. Find all optimizers of

$$f(x, y) = xy + x + y$$

subject to the constraint $x^2y^2 = 4$.

Chapter 3

Integration

3.1 Double Integrals

Let $D \subset \mathbb{R}^2$ be a bounded region with piecewise smooth boundary. Let $R = [a, b] \times [c, d]$ be any rectangle containing D. Suppose $f : D \to \mathbb{R}$ is a bounded function and extend the definition of f to all of R be requiring $f(x) = 0$ for $x \in R \setminus D$. Consider a 'partition' $P_{m,n}$ of R consisting of subrectangles generated by the partitions

$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b \qquad \text{and} \qquad c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$$

of $[a, b]$ and $[c, d]$ respectively, where

$$x_i = x_0 + i\frac{b-a}{m} \qquad i = 1, 2, 3, \cdots, m$$

and

$$y_j = y_0 + j\frac{d-c}{n} \qquad j = 1, 2, 3, \cdots, n.$$

This partition consists of mn rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Set $\Delta x = (b-a)/n$ and $\Delta y = (d-c)/n$ so that $\Delta A = \Delta x \Delta y$ is the area of each sub rectangle R_{ij}. Here is a picture of the situation:

Definition 3.1.1. The *Riemann sum* of f corresponding to the partition $P_{m,n}$ of R and the sample points $(x_i^*, y_j^*) \in R_{ij}$ is

$$\mathscr{R}(f, P_{m,n}) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_i^*)\Delta A.$$

Definition 3.1.2. Let $f : D \to \mathbb{R}$ and let R be any rectangle containing D. Extend the definition of f to be zero on $R \setminus D$. The *double integral* of f over D is

$$\int\int_D f(x, y)\, dA = \lim_{m,n\to\infty} \mathscr{R}(f, P_{m,n}) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A,$$

provided the limit exists. If the limit exists, we say f is *integrable* on D.

Remark 3.1.3. 'Most' functions are not integrable, but all functions you will see in this course are integrable.

Remark 3.1.4. If $f(x, y) \geq 0$ for $(x, y) \in D$, then when it exists, $\int\int_D f(x, y)\, dA$ is the volume of the solid bounded below by D and bounded above by the graph of f.

Remark 3.1.5. If f is integrable on D, then $\int\int_D f(x, y)\, dA$ can be approximated (with arbitrarily fine precision) by an appropriate Riemann sum. Common choices for representative points are

1. Midpoints

$$(x_i^*, y_j^*) = \left(\frac{x_{i-1} + x_i}{2}, \frac{y_{j-1} + y_j}{2}\right), \qquad i = 1, \cdots, m;\ j = 1, \cdots, n.$$

2. Lower-left corners

$$(x_i^*, y_j^*) = (x_{i-1}, y_{j-1}), \qquad i = 1, \cdots, m;\ j = 1, \cdots, n.$$

3. Lower-right corners

$$(x_i^*, y_j^*) = (x_i, y_{j-1}), \qquad i = 1, \cdots, m;\ j = 1, \cdots, n.$$

4. etc.

Example 3.1.6. Let $f(x, y) = x + y^2$ and let $D = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 4\}$. Use a Riemann sum with four equally-sized subrectangles and lower-right corners as representative points to approximate $\int\int_D f(x, y)\, dA$.

Solution

Proposition 3.1.7 (Properties of Double Integrals). *Suppose f and g are integrable on a region $D \subset \mathbb{R}^2$ and c is a constant.*

1. *Linearity:*

$$\int\int_D (cf(x,y) + g(x,y))\, dA = c\int\int_D f(x,y)\, dA + \int\int_D g(x,y)\, dA.$$

2. *Area of Domain: If A is the area of D, then*

$$A = \int\int_D dA.$$

3. *Additivity: If D_1 and D_2 are disjoint subsets of D such that $D = D_1 \cup D_2$, then*

$$\int\int_D f(x,y)\, dA = \int\int_{D_1} f(x,y)\, dA + \int\int_{D_2} f(x,y)\, dA.$$

4. *Order Preserving: If $f(x,y) \leq g(x,y)$ for all $(x,y) \in D$, then*

$$\int\int_D f(x,y)\, dA \leq \int\int_D g(x,y)\, dA.$$

The following theorem is among the most useful for evaluating integrals.

Theorem 3.1.8 (Fubini's Theorem). *If f is continuous on the rectangle $[a,b] \times [c,d]$ then*

$$\int\int_D f(x,y)\, dA = \int_a^b \left(\int_c^d f(x,y)\, dy\right)\, dx = \int_c^d \left(\int_a^b f(x,y)\, dx\right)\, dy.$$

Example 3.1.9. Evaluate $\int\int_D f(x,y)\,dA$ if $f(x,y) = \sin(x+y)$ and $D = [0,\pi] \times [-\pi/2, \pi/2]$.

Solution

The following example shows a convenient way of estimating integrals of functions whose maximum and minimum values over a specified region are known.

Example 3.1.10. Let $f(x, y) = x^2+y^2+7$ and let R be the rectangle $[-1, 1]\times[-1, 1]$. It is easy to show that $\max_{x\in R} f(x) = 9$ and $\min_{x\in R} f(x) = 4$. Given these maximum and minimum values for f, use the properties of integrals in Proposition 3.1.7 to compute upper and lower bounds for $\int\int_R f(x, y)\, dA$.

Solution

Definition 3.1.11. Suppose f is integrable on D and let A denote the area of D. The *average value* of f over D is

$$\mathcal{A}f = \frac{1}{A} \int\int_D f(x, y)\, dA.$$

Example 3.1.12. Compute the average value of $f(x, y) = 5 - xy$ over the rectangular region $D = [-2, 1] \times [1, 3]$.

Solution

3.1.1 Integration Over General Regions

Definition 3.1.13. 1. A region $D \subset \mathbb{R}^2$ is called *vertically simple* if there are numbers $a < b$ and functions $y_1(x) \leq y_2(x)$ such that

$$D = \{(x, y) : a \leq x \leq b; y_1(x) \leq y \leq y_2(x)\}. \tag{3.1}$$

2. A region $D \subset \mathbb{R}^2$ is called *horizontally simple* if there are numbers $c < d$ and functions $x_1(y) \leq x_2(y)$ such that

$$D = \{(x, y) : x_1(y) \leq x \leq x_2(y); c \leq y \leq d\}. \tag{3.2}$$

Example 3.1.14. Let D be the region in the xy-plane bonded by $x^2 + y^2 = 1$ such that $y \geq 0$. This region is both horizontally and vertically simple. Provide a description of D as a horizontally simple region and a description of D as a vertically simple region.

Solution

The following proposition is similar to Fubini's Theorem and will be very useful for evaluating integrals over simple regions.

Proposition 3.1.15 (Evaluating Integrals Over Simple Regions). *Suppose f is integrable on the region $D \subset \mathbb{R}^2$.*

1. *If D is vertically simple with description as in equation (3.1), then*

$$\int\int_D f(x,y)\ dA = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\ dy\ dx.$$

2. *If D is horizontally simple with description as in equation (3.2), then*

$$\int\int_D f(x,y)\ dA = \int_c^d \int_{x_1(y)}^{x_2(y)} f(x,y)\ dx\ dy.$$

Remark 3.1.16. If D is neither vertically simple nor horizontally simple, we may still be able to compute $\int\int_D f(x,y)\ dA$. If possible, simply cut D into pieces, where each piece is either vertically simple or horizontally simple. By additivity of the double integral, we can integrate f over each of these pieces then add the results of the integrals to recover $\int\int_D f(x,y)\ dA$.

Example 3.1.17. Let D be the region in the xy-plane bounded by the curves $y = x^2$ and $y = 2$ and let $f(x, y) = 3x^2y$.

1. Evaluate $\int\int_D f(x, y)\, dA$ by viewing D as a vertically simple region.
2. Evaluate $\int\int_D f(x, y)\, dA$ by viewing D as a horizontally simple region.
3. Compute the average value of f over D.

Solution

Example 3.1.18. Let D be the region in the xy-plane bounded by the curves $x - 4y = 20$ and $x = y^2 - 1$. Let V be the volume of the solid which is bounded below by $z = 0$, bounded above by $z = 4x + 1$ and corresponding to $(x, y) \in D$. Compute V in two different ways; by viewing D as a vertically simple region and by viewing D as a horizontally simple region. Which is easier?

Solution

For some integrals, only one order of integration is possible.

Example 3.1.19. Let D be the region in the xy-plane bounded by $x = 0$, $y = \sqrt{\pi}$ and $y = x$. Evaluate $\int\int_D \sin(y^2)\, dA$.

Solution

3.1.2 Exercises

1. Let $D = \{(x, y) : 0 \le x \le \pi, 0 \le y \le \pi\}$. Use a Riemann sum whose partition consists of 9 equally sized rectangles and whose representative points are upper-left corners to approximate $\int\int_D \sin(x + y)\, dA$.

2. Use the order-preserving property of double integrals to find upper and lower bounds for $\int\int_D f(x, y)\, dA$.

   (a) $f(x, y) = xy^3$, $D = \{(x, y) : 1 \le x \le 2, 1 \le y \le 2\}$.

   (b) $f(x, y) = \sqrt{2 + xe^{-y}}$, $D = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$.

   (c) $f(x, y) = x^2 + 4y^2 + 3$, $D = \{(x, y) : x^2 + 4y^2 \le 1\}$.

   (d) $f(x, y) = 2x^2 + y^2 + 2x - 3x$, $D = \{(x, y) : x^2 + y^2 \le 1\}$.

3. Evaluate the integrals over the specified regions.

   (a) $\int\int_D xy\, dA$, D is the region bounded by $y = x$ and $y = x^2$.

   (b) $\int\int_D xy^2\, dA$, D is bounded by $y = 2 + x^2$ and $y = 4 - x^2$.

   (c) $\int\int_D \sqrt{1 - y^2}\, dA$, D is the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$.

   (d) $\int\int_D (1 + y)\, dA$, D is bounded by $x = 3$ and $x + y^2 = 4$.

4. Find the volume of the solid E.

   (a) E is bounded by the plane $x + y + z = 2$ and the coordinate planes.

   (b) E is the solid under the paraboloid $z = 2x^2 + y^2$ and above the region in the xy-plane bounded by $x = y^2$ and $x = 1$.

   (c) E is the solid in the first octant bounded by the cylinder $x^2 + y^2 = 1$ and the planes $x = 0$, $z = 0$ and $y = z$.

5. Sketch the region of integration and reverse the order of integration.

   (a) $\int 1^4 \int_{\sqrt{y}}^{2} f(x, y)\, dx\, dy$

   (b) $\int_0^1 \int_{x^3}^{\sqrt{x}} f(x, y)\, dy\, dx$

   (c) $\int_0^1 \int_1^{e^y} f(x, y)\, dx\, dy$

   (d) $\int_0^3 \int_0^{y/3} f(x, y)\, dx\, dy + \int_3^6 \int_0^{6-y} f(x, y)\, dx\, dy$

   (e) $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} f(x, y)\, dy\, dx$

6. Reverse the order of integration then evaluate the integral

(a) $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\, dx\, dy$
Hint: After reversing the order of integration, use the substitution $u = x^2$.

(b) $\int_0^4 \int_{\sqrt{x}}^2 (1 + y^3)^{-1}\, dy\, dx$.

3.2 Double Integrals in Polar Coordinates

Recall the area ΔA of a sector of radius r and angle $\Delta\theta$:

By subtracting the area of a sector of radius r and angle $\Delta\theta$ from the area of a sector of radius $r + \Delta r$ and angle $\Delta\theta$, we obtain the area ΔA of the annular sector of inner radius r and outer radius $r + \Delta r$ (for $\Delta r > 0$):

If Δr is very small, then $(\Delta r)^2 \ll \Delta r$, so

$$\Delta A \approx$$

The *polar area element* is

$$dA = r\ dr\ d\theta = r\ dA',$$

where $dA' = dr\ d\theta$.

Recall the definition of polar coordinates

$$\begin{aligned} x &= r\cos\theta \qquad & r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta \qquad & \theta &= \arctan\left(\tfrac{y}{x}\right). \end{aligned}$$

Proposition 3.2.1 (Integration in Polar Coordinates). *Suppose f is an integrable function of (x, y) for $(x, y) \in D \subset \mathbb{R}^2$ and suppose D is the image of another region $D' \subset \mathbb{R}^2$ under the transformation $T : D' \to D$ given by*

$$T(r, \theta) = (x(r, \theta), y(r, \theta)) = (r\cos\theta, r\sin\theta),$$

Then

$$\int\int_D f(x, y)\ dA = \int\int_{D'} f(r\cos\theta, r\sin\theta)\ r\ dA',$$

where $dA = dx\ dy$ is the rectangular area element and $r\ dA' = r\ dr\ d\theta$ is the polar area element.

Example 3.2.2. Let D be the portion of the disk $x^2 + y^2 \leq 1$ in the first quadrant. Compute

$$\int\int_D xy^2\sqrt{x^2+y^2}\ dA.$$

Solution

When converting a rectangular integral to a polar integral, use the following three steps.

1. Find $D' = \{(r, \theta) : (r\cos\theta, r\sin\theta) \in D\}$.
2. Write the integral in polar coordinates by replacing each instance of x with $r\cos\theta$ and each instance of y with $r\sin\theta$. Remember to replace the rectangular area element $dA(x, y)$ with the polar area element $r\,dA'(r, \theta) = r\,dr\,d\theta$.
3. Evaluate the integral in terms of r and θ over D', the new integration domain.

Remark 3.2.3. The following cases suggest that a conversion to polar coordinates may simplify the evaluation of an integral.

1. If the integration region D is bounded by circles centered at the origin, lines through the origin or polar graphs $r = f(\theta)$.
2. The integrand depends on x and y via the combinations $x^2 + y^2$ or y/x, which correspond to r^2 and $\tan\theta$ respectively.

Example 3.2.4. Let D be the region in the first quadrant bounded by $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$. Compute $\int\int_D xy \, dA$.

Solution

Example 3.2.5. Compute the volume of the solid bounded by the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = 2 - x^2 - y^2$ with $y \geq 0$.

Solution

Example 3.2.6. Use a double integral to compute the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $(x-1)^2 + y^2 = 1$ corresponding to $0 \leq x \leq y$.

Solution

3.2.1 Exercises

1. Convert the double integral $\int\int_D f(x,y)\ dA(x,y)$ to an iterated integral in polar coordinates.

   (a) $D = \{(x,y) : x^2+y^2 \le R^2\}$, where R is a positive constant.

   (b) $D = \{(x,y) : x^2+y^2 \le 3x\}$.

   (c) $D = \{(x,y) : 1/4 \le x^2+y^2 \le 16,\ y \ge 0\}$

2. Evaluate the double integral by changing to polar coordinates

   (a) $\int\int_D xy\ dA$, $D = \{(x,y) : 1 \le x^2+y^2 \le 4\}$

   (b) $\int\int_D \sin(x^2+y^2)\ dA$, $D = \{(x,y) : x^2+y^2 \le 25\}$

   (c) $\int\int_D \arctan(y/x)\ dA$, $D = \{(x,y) : 1 \le x^2+y^2 \le 9 \text{ and } x/\sqrt{3} \le y \le \sqrt{3}x\}$

3. Sketch the region of integration. Evaluate the integral by converting to polar coordinates.

   (a) $\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}(1+x^2+y^2)^{-1}\ dx\ dy$

   (b) $\int_0^3\int_0^{\sqrt{9-y^2}}(9-x^2-y^2)^{-1/2}\ dx\ dy$

   (c) $\int_0^4\int_x^4 x^2\ dy\ dx$

   (d) $\int_{-1}^{1}\int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\ dx\ dy$

   (e) $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(x+y)\ dy\ dx$

   (f) $\int_0^2\int_0^{\sqrt{2y-y^2}} \sqrt{x^2+y^2}\ dx\ dy$

4. Use an appropriate double integral to compute the area of the region D.

   (a) D is the region bounded by the spirals $r = \theta/2$ and $r = \theta$ for $0 \le \theta \le 2\pi$ and the positive x-axis.

   (b) D is the region enclosed by the polar graph $r = 1+\cos\theta$.

   (c) D is the part of the cardioid $r = 1+\cos\theta$ that lies in quadrant I.

   (d) D is the part of the cardioid $r = 1+\cos\theta$ that lies in quadrant II.

   (e) D is the region inside the smaller loop of $r = 1-2\cos\theta$.

   (f) D is the part of the region enclosed by the cardioid $r = 1+\sin\theta$ that lies outside of the disk $x^2+y^2 \le 9/4$.

(g) D is the region in the first quadrant bounded by the curve $(x^3 + y^3)^2 = x^2 + y^2$.

5. Compute the average value of the function $f(x, y)$ over the indicated region D.

   (a) $f(x, y) = (x^2 + y^2)^{3/2}$, $D = \{(x, y) : 0 \le x \le \sqrt{1 - y^2},\ 0 \le y \le 1\}$.
   (b) $f(x, y) = \sin(x^2 + y^2)$, $D = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le \sqrt{1 - x^2}\}$.
   (c) $f(x, y) = (x^2 + y^2)^{-1/2}$, $D = \{(x, y) : 1 \le y \le 2,\ 0 \le x \le \sqrt{2y - y^2}\}$.

6. Compute the volume of the solid E.

   (a) E is bounded by the cones $z = 3\sqrt{x^2 + y^2}$ and $z = 4 - \sqrt{x^2 + y^2}$.
   (b) E is bounded by the cone $z = \sqrt{x^2 + y^2}$, the plane $z = 0$ and the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
   (c) E is bounded by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = -3$.
   (d) E is bounded by the hyperboloid $x^2 + y^2 - z^2 = 1$ and the planes $z = 0$, $z = 2$.
   (e) E is the region under the paraboloid $z = x^2 + y^2$ above the xy-plane and inside the cylinder $x^2 + y^2 = 2x$.

3.3 Change of Variables in Double Integrals

Definition 3.3.1. Let D' be an open subset of the uv-plane and let $T : D' \to \mathbb{R}^2$ be a transformation that takes (u, v) to (x, y) (i.e. T expresses x and y as functions of u and v)

$$T(u, v) = (x(u, v), y(u, v)).$$

Suppose the component functions $x(u, v)$ and $y(u, v)$ of T have continuous partial derivatives. The *Jacobian* $J_T(u, v)$ of T is

$$J_T(u, v) = \det \begin{pmatrix} \frac{\partial x}{\partial u}(u, v) & \frac{\partial x}{\partial v}(u, v) \\ \frac{\partial y}{\partial u}(u, v) & \frac{\partial y}{\partial v}(u, v) \end{pmatrix}$$

Example 3.3.2. Let $T : [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2$ be given by

$$T(r, \theta) = (r \cos \theta, r \sin \theta),$$

(so T is the usual rectangular-to-polar coordinate transformation that expresses x and y as $x = r \cos \theta$ and $y = r \sin \theta$). Compute the Jacobian of T.

Solution

Theorem 3.3.3 (Change of Variable for Double Integrals). *Suppose D' and D are bounded subdomains of $\mathbb{R}^2$ with piecewise smooth boundaries and that $T : D' \to D$ is a transformation whose domain is D' and whose image is D. Suppose the component functions $x(u,v)$ and $y(u,v)$ of T have continuous partial derivatives with respect to u and v, and that $J_T(u,v)$ is nonzero except possibly on the boundary of D'. If $f : D \to \mathbb{R}$ is a continuous function, then*

$$\int\int_D f(x,y)\, dA(x,y) = \int\int_{D'} f(x(u,v), y(u,v))\ |J_T(u,v)|\, dA'(u,v), \tag{3.3}$$

where $dA(x,y)$ is the rectangular area element and $dA'(u,v)$ is the uv-area element.

Remark 3.3.4. Here are steps for changing variables in double integrals. Given that we want to compute $\int\int_D f(x,y)\, dA$ (where D and f are given), execute the following steps

1. Find a transformation $T(u,v) = (x(u,v), y(u,v))$ and a set D' such that the given integration domain D is the image of D' under T. A useful thing to remember for this step is

$$(\text{ boundaries of } D') \overset{T}{\mapsto} (\text{ boundaries of } D).$$

2. Replace all instances of x and y in f's formula with the new variables u and v according to $x(u,v)$ and $y(u,v)$ as prescribed by T. That is $f(x,y)$ gets replaced by $f(x(u,v), y(u,v))$.

3. Compute the Jacobian $J_T(u,v)$ and the area element $dA(x,y) = |J_T(u,v)|\, dA'(u,v)$.

4. Evaluate $\int\int_D f(x,y)\, dA(x,y)$ according to equation (3.3).

Example 3.3.5. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$. Compute the average distance from $(0, 0)$ to a generic point in D.

Solution

Example 3.3.6. Let a and b be positive constants. Compute the area of the region bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution

Theorem 3.3.7 (Inverse Function Theorem). *Suppose D' is an open subset of the uv-plane and $T(u,v) = (x(u,v), y(u,v))$ is a transformation of D' whose component functions $x(u,v)$ and $y(u,v)$ have continuous first-order partials in D'. If $(u_0, v_0) \in D'$ and $J_T(u_0, v_0) \neq 0$, then there in an open neighborhood D of $(x(u_0,v_0), y(u_0,v_0)) = (x_0, y_0)$ on which an inverse transformation $T^{-1}(x,y) = (u(x,y), v(x,y))$ exists. Moreover, T^{-1} has continuous first-order partials in D and*

$$J_T(u,v) = \frac{1}{J_{T^{-1}}(x(u,v), y(u,v))}. \tag{3.4}$$

Example 3.3.8. Let D be the region bounded by the lines $y = x$, $y = 4x$ and the hyperbolas $xy = 1$, $xy = 2$. Compute $\int\int_D xy^3 \, dA$.

Solution

Remark 3.3.9. As in the previous example, computing $J_T(u,v)$ from $u(x,y)$ and $v(x,y)$ may not require finding $x(u,v)$ or $y(u,v)$. This is (part of) the power of the Inverse Function Theorem (see equation (3.4)).

Example 3.3.10. Let $D = \{(x,y) : |x| + |y| \leq 1\}$. Use an appropriate change of variables to evaluate $\int\int_D e^{x-y}\, dA$.

Solution

3.3.1 Exercises

1. Use the given change of variables to evaluate $\int\int_D f(x, y)\, dA$.

   (a) $f(x, y) = 2x + y$. D is the parallelogram with vertices $(3, -1)$, $(-3, 1)$, $(5, 1)$ and $(-1, 3)$. The change of variable is $x = (v - 3u)/4$, $y = (u + v)/4$.

   (b) $f(x, y) = x^2 - xy + y^2$. D is the region bounded by the ellipse $x^2 - xy + y^2 = 1$. The change of variable is $x = (u - v)/\sqrt{3}$, $y = (u + v)/\sqrt{3}$.

2. Use an appropriate change of variable to evaluate $\int\int_D f(x, y)\, dA$.

   (a) $f(x, y) = xy^2$. D is the region in quadrant I bounded by $xy = 1$, $xy = 2$, $xy^2 = 1$ and $xy^2 = 2$.

   (b) $f(x, y) = 1 + 3x^2$. D is the region bounded by the lines $x + y = 1$, $x + y = 3$ and the curves $y = x^3$, $y = x^3 + 1$.

   (c) $f(x, y) = 3xy$. D is the region bounded by the lines $x = 2y$, $x = 2y - 4$, $x + y = 4$ and $x + y = 1$.

3. Compute the average value of $f(x, y)$ over the region D.

   (a) $f(x, y) = (x + y)^2 \sin^2(x - y)$. D is bounded by the lines $x + y = 1$, $x - y = 1$, $x + y = 3$ and $x - y = -1$.

   (b) $f(x, y) = (3x + 2y)(2y - x)^{3/2}$. D is the parallelogram with vertices $(0, 0)$, $(-2, 3)$, $(2, 5)$ and $(4, 2)$.

   (c) $f(x, y) = \sqrt{x + y}$. D is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.

4. Compute the area of the region in the first quadrant bounded by the lines $y = x$, $y = 2x$ and the hyperbolas $xy = 1$ and $xy = 2$.

5. Compute the area of the region outside the circle $r = 1$ and inside the cardioid $r = 1 + \cos\theta$.

6. Evaluate $\int\int_D \sqrt{4 - x^2 - y^2}\, dA(x, y)$, where D is the region bounded by the loop of the lemniscate $(x^2 + y^2)^2 = 4(x^2 - y^2)$ and corresponding to $x \geq 0$.

7. Compute the volume of the solid bounded by the plane $z = 0$, the cone $z = \sqrt{x^2 + y^2}$ and the cylinder $x^2 + y^2 = 4x$.

8. Compute the volume of the solid region bounded by $z = x^2 + y^2$ and the elliptic cylinder
$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$
corresponding to $z \geq 0$.

9. Evaluate
$$\int\int_D \cos\left(\frac{x-y}{x+y}\right)\, dA,$$
where D is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,1)$ and $(0,2)$.

10. Compute the volume of the solid bounded below by $z=0$ and bounded above by $f(x,y)$ for $(x,y)\in D$.

    (a) $f(x,y)=(x+y)e^{x-y}$. D is the square with vertices $(4,0)$, $(6,2)$, $(4,4)$ and $(2,2)$.

    (b) $f(x,y)=\sqrt{(y-x)(y+4x)}$. D is the parallelogram with vertices $(0,0)$, $(1,1)$, $(0,5)$ and $(-1,4)$.

3.4 Center of Mass and Moments of Inertia

Definition 3.4.1. Let $\sigma(x, y)$ be a continuous mass density function defined on a planar region D.

1. The *mass* m of D is
$$m = \int\int_D \sigma(x, y)\, dA.$$

2. The *center of mass* of D is the ordered pair $(\overline{x}, \overline{y})$, where
$$\overline{x} = \frac{1}{m}\int\int_D x\sigma(x, y)\, dA \qquad \text{and} \qquad \overline{y} = \frac{1}{m}\int\int_D y\sigma(x, y)\, dA.$$

3. The *second moment* or *moment of inertia* of D relative to the x-axis is
$$I_x = \int\int_D \underbrace{y^2}_{\text{(square distance to } x \text{ axis)}} \sigma(x, y)\, dA.$$

   The *second moment* or *moment of inertia* of D relative to the y-axis is
$$I_y = \int\int_D \underbrace{x^2}_{\text{(square distance to } y \text{ axis)}} \sigma(x, y)\, dA.$$

Example 3.4.2. Let D be the region bounded by $y = 0$ and $y = 4 - x^2$ and suppose the mass density of D at (x, y) is proportional to the distance from (x, y) to the x-axis.

1. Compute the center of mass of D.

2. Compute the moment of inertia of D about the x-axis.

Solution

3.4.1 Exercises

1. Compute the mass of D.

    (a) D is the portion of the circle $x^2 + y^2 = 9$ in quadrant I with mass density at (x, y) which is proportional to the distance from (x, y) to the origin.

    (b) $D = \{(x, y) : x \geq 0,\ 3 \leq y \leq 3 + \sqrt{9 - x^2}$. The mass density of D is $\sigma(x, y) = xy$.

2. Compute the mass and center of mass of D.

    (a) D is the rectangle with vertices $(0, 0)$, $(2, 0)$, $(0, 3)$ and $(2, 3)$. The mass density of D at (x, y) is proportional to the distance from (x, y) to the y-axis.

    (b) D is the rectangle with vertices $(0, 0)$, $(3, 0)$, $(0, 1)$ and $(3, 1)$. The mass density of D at (x, y) is proportional to the square-distance from (x, y) to the origin.

    (c) D is bounded by $y = 0$ and $y = \sqrt{9 - x^2}$. The mass density of D is $\sigma(x, y) = k(3 - y)y$, where k is a positive constant.

    (d) D is bounded by $y = 0$, $x = -1$, $x = 1$ and $y = (1 + x^2)^{-1}$. The mass density of D is constant.

    (e) D is bounded by $y = 9 - x^2$ and $y = 0$. The mass density of D at (x, y) is proportional to the square-distance from (x, y) to the x-axis.

3. Compute the moment of inertia about the x-axis of a rectangle of constant mass density whose vertices are $(0, 0)$, $(b, 0)$, (b, h) and $(0, h)$, where b and h are positive constants.

4. Compute the moment of inertia about the y-axis of a right triangle with vertices $(0, 0)$, $(0, h)$ and $(b, 0)$ if the mass density of the triangle is constant. Here b and h are positive constants.

3.5 Triple Integrals

Definition 3.5.1. Let $E \subset \mathbb{R}^3$ be a bounded solid region and let $f : E \to \mathbb{R}$. The *triple integral* of f over E is

$$\begin{aligned}\int\int\int_E f(x,y,z)\,dV &= \lim_{\ell,m,n\to\infty} \mathscr{R}(f, P_{\ell,m,n}) \\ &= \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*)\,\underbrace{\Delta x\;\Delta y\;\Delta z}_{\Delta V},\end{aligned}$$

provided the limit exists, where $P_{\ell,m,n}$ is an appropriate partition of a cube containing E. If the limit exists, f is said to be *integrable* on E.

Remark 3.5.2. Since the definition of triple integrals is similar to the definition of double integrals, many of the details are omitted.

Proposition 3.5.3 (Properties of Triple Integrals). *Let f and g be integrable functions on a solid region $E \subset \mathbb{R}^3$ and let c be a constant. The following properties hold.*

1. *Linearity.*

$$\int\int\int_E (cf(x,y,z)+g(x,y,z))\,dV = c\iiint_E f(x,y,z)\,dV + \iiint_E g(x,y,z)\,dV.$$

2. *Additivity. If $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, then*

$$\iiint_E f(x,y,z)\,dV = \iiint_{E_1} f(x,y,z)\,dV + \iiint_{E_2} f(x,y,z)\,dV.$$

3. *Volume.*

$$\text{Volume of } E = \iiint_E dV.$$

4. *Order Preserving. If $f(x,y,z) \le g(x,y,z)$ for $(x,y,z) \in E$, then*

$$\iiint_E f(x,y,z)\,dV \le \iiint_E g(x,y,z)\,dV.$$

Proposition 3.5.4 (Evaluating Triple Integrals). *If E admits the description*

$$E = \{(x, y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x),\ h_1(x, y) \leq z \leq h_2(x, y)\},$$

for some continuous functions g_1, g_2, h_1, h_2 and if f is continuous on E then

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\, dz\, dx\, dy.$$

Evaluating triple integrals relative to other orders of dx, dy and dz is similar.

Example 3.5.5. Evaluate $\int_0^3 \int_0^x \int_0^{x+y} e^x(y + 2z)\, dz\, dy\, dx$.

Solution

Example 3.5.6. Let E be the region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. Compute the volume of E.

Solution

Example 3.5.7. Evaluate the triple integral

$$\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin y^2 \; dz \; dy \; dx.$$

What do you notice about the given order of integration?

Solution

Definition 3.5.8. Let E be a solid region in $\mathbb{R}^3$ with mass density function $\sigma(x,y,z)$.

1. The *mass* m of E is
$$m = \iiint_E \sigma(x,y,z)\,dV.$$
2. The *center of mass* of E is the ordered triple $(\overline{x},\overline{y},\overline{z})$, where
$$\begin{aligned}\overline{x} &= \frac{1}{m}\iiint_E x\sigma(x,y,z)\,dV,\\ \overline{y} &= \frac{1}{m}\iiint_E y\sigma(x,y,z)\,dV,\\ \overline{z} &= \frac{1}{m}\iiint_E z\sigma(x,y,z)\,dV.\end{aligned}$$
3. The *second moments* or *moments of inertia* of D about the coordinate axes are
$$\begin{aligned}I_x &= \iiint_E \underbrace{(y^2+z^2)}_{\text{(square distance to } x\text{-axis)}} \sigma(x,y,z)\,dV && \text{(about } x\text{-axis)}\\ I_y &= \iiint_E \underbrace{(x^2+z^2)}_{\text{(square distance to } y\text{-axis)}} \sigma(x,y,z)\,dV && \text{(about } y\text{-axis)}\\ I_z &= \iiint_E \underbrace{(x^2+y^2)}_{\text{(square distance to } z\text{-axis)}} \sigma(x,y,z)\,dV && \text{(about } z\text{-axis)}\end{aligned}$$

Example 3.5.9. Find the x-coordinate of the center of mass of the cube of side length 1 in octant I with one vertex at the origin if the mass density at a point (x, y, z) is proportional to the square distance from (x, y, z) to the origin.

Solution

Example 3.5.10. Let E be the region bounded below by $z = 0$ and bounded above by $x^2 + y^2 + z^2 = 4$. Compute the moment of inertia of E about the x-axis if the mass density of E at (x, y, z) is proportional to the distance from (x, y, z) to the plane $z = 0$.

Solution

3.5.1 Exercises

1. Set up a triple integral to determine the volume of the solid E.

   (a) E is the first-octant portion of the solid bounded by the planes $x+y=1$, $x+y=3$ and the cylinder $z=1-y^2$.

   (b) E is the solid bounded above by $z=\sqrt{1-x^2-y^2}$ and bounded below by $z=0$.

   (c) E is the solid bounded below by $z=x^2+y^2$ and bounded above by $x^2+y^2+z^2=6$.

2. Use a triple integral to compute the volume of the solid E.

   (a) E is bounded by $x+y^2=4$, $z=0$ and $z=x$.

   (b) E is bounded by $z=0$, $z=xy$, $x=0$, $x=1$ $y=0$ and $y=1$.

   (c) E is the solid in the first octant bounded by $z=4-x^2$ and $y=4-x^2$.

3. Rewrite the given triple integral with the indicated order of integration.

   (a) $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{6-x-y} \, dz \, dy \, dx$. Use the order $dz \, dx \, dy$.

   (b) $\int_0^2 \int_{2x}^4 \int_0^{\sqrt{y^2-4x^2}} \, dz \, dy \, dx$. Use the order $dx \, dy \, dz$.

4. Let E be the solid bounded by $x=0$, $y=0$, $z=0$ and $3x+3y+5z=15$ and suppose the mass density of E at (x,y,z) is proportional to the distance from (x,y,z) to the xz-plane. Compute the y-coordinate of the center of mass of E.

3.6 Triple Integrals in Cylindrical and Spherical Coordinates

Definition 3.6.1. Cylindrical coordinates (r, θ, z) are a three-dimensional coordinate system, where the xy-plane is polar coordinates and the z-coordinate is unchanged. The rectangular-to-cylindrical coordinate transformation is

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z, \end{cases} \qquad \text{where} \qquad 0 \le r < \infty,\ 0 \le \theta < 2\pi,\ -\infty < z < \infty. \tag{3.5}$$

Proposition 3.6.2. *The rectangular volume element dV is expressed in terms of cylindrical coordinates as*

$$dV = r\ dV',$$

where $dV = dx\ dy\ dz$ (or any permutation of dx, dy and dz) and $dV' = dr\ d\theta\ dz$ (or any permutation of dr, $d\theta$ and dz).

Example 3.6.3. Find the solid region E' whose image under the rectangular-to-cylindrical coordinate transformation is the solid E in the first octant bounded by the paraboloid $z = x^2 + y^2$ and the planes $z = 4$, $y = x$ and $y = 0$.

Solution

Proposition 3.6.4. *If f is integrable on a solid region E, then*

$$\iiint_E f(x,y,z)\ dV = \iiint_{E'} f(r\cos\theta, r\sin\theta, z)\ r\ dV',$$

where $dV' = dr\ d\theta\ dz$ (in an appropriate order) and E' is a cylindrical-coordinate description of E (i.e. E' is the preimage of E under the coordinate transformation in equation (3.5)*).*

Example 3.6.5. Compute the volume of the solid E which lies inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $r = 2\cos\theta$ in two different ways.

1. Use a double integral to integrate a suitable function of the form $f_{\text{upper}} - f_{\text{lower}}$.
2. Use a triple integral in cylindrical coordinates.

Solution

Example 3.6.6. Let $E = \{(x, y, z) : x^2 + y^2 \leq 4,\ 0 \leq z \leq 1\}$.

1. Sketch E.
2. Let f be a continuous function. Simplify $\iiint_E f(x^2+y^2)\, dV$ to a single integral whose integration domain is an interval.

Solution

Example 3.6.7. Let E be the solid region bounded by the ellipsoid $4x^2+4y^2+z^2 = 16$ and the xy-plane corresponding to $z \geq 0$. Suppose the mass density of E at a point (x, y, z) is proportional to the distance from (x, y, z) to the xy-plane. Compute the mass of E.

Solution

Example 3.6.8. Let E be the solid bounded by $z = x^2 + y^2$ and $z = 4$ and suppose the mass density of E at a point (x, y, z) is proportional to the distance from (x, y, z) to the z-axis. Compute the moment of inertia of E about the z-axis in two different ways; by integrating dz first and by integrating dr first.

Solution

Definition 3.6.9. The spherical coordinates (ρ, θ, ϕ) are a three dimensional coordinate system whose relation to rectangular coordinates is

$$(3.6)\qquad \begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi, \end{cases} \qquad \text{where} \qquad 0 \le \rho < \infty,\ 0 \le \phi \le \pi,\ 0 \le \theta < 2\pi.$$

Proposition 3.6.10. *The expression of the rectangular volume element dV in terms of spherical coordinates is*

$$dV = \rho^2 \sin\phi \, dV',$$

where dV is any permutation of dx, dy, dz and dV' is any permutation of dr, $d\theta$, $d\phi$.

Proposition 3.6.11. *Let f be an integrable function on the solid region E. Then*

$$\iiint_E f(x, y, z)\, dV = \iiint_{E'} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, dV',$$

where $dV' = d\rho\ d\phi\ d\theta$ (in an appropriate order) and E' is a spherical-coordinate description of E (i.e. E' is the preimage of E under the coordinate transformation given in (3.6)*).*

Example 3.6.12. Let E be the part of the solid bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the cone $z^2 = 3(x^2 + y^2)$ that lies in the first octant. Find the region E' whose image under the rectangular-to-shperical coordinate transformation (equations (3.6)) is E.

Solution

Example 3.6.13. Compute the volume of the solid E bounded by $x^2 + y^2 + z^2 = 9$ corresponding to $z \geq \sqrt{3(x^2 + y^2)}$.

Solution

Example 3.6.14. Compute the moment of inertia about the z-axis of the solid bounded below by $z = \sqrt{x^2 + y^2}$ and above by the upper half of the sphere $x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}$ if the mass density at (x, y, z) is inversely proportional to the square distance from (x, y, z) to the xy-plane.

Solution

Example 3.6.15. Let E be the solid bounded by the spheres $x^2+y^2+z^2=R_1^2$ and $x^2+y^2+z^2=R_2^2$ corresponding the $z\geq 0$, where $0<R_1\leq R_2$. If E has uniform mass density, compute the center of mass of E.

Solution

3.6.1 Exercises

1. For each given solid E, find the region E' whose image under the rectangular-to-cylindrical coordinate transformation (equations (3.5)) is E.
   (a) E is bounded by the cylinder $x^2+y^2=4$, the paraboloid $z=x^2+y^2$ and the plane $z=0$.
   (b) E is bounded by the cone $(z+1)^2=x^2+y^2$and the cylinder $x^2+y^2=9$.
   (c) E id bounded by the paraboloid $z=x^2+y^2$, the cylinder $x^2+y^2=4x$ and the plane $z=0$.
   (d) E is the part of the ball $x^2+y^2+z^2 \leq 16$ in the first octant.
2. Compute the volume of the solid bounded by $0 \leq r \leq 4\sqrt{\sin(2\theta)}$ for $0 \leq z \leq r^2$ and $0 \leq \theta \leq \pi/2$.
3. Evaluate the integral
$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_0^3 z\sqrt{x^2+y^2}\ dz\ dy\ dx$$
   by converting to cylindrical coordinates.
4. Compute the mass of the solid $E=\{(x,y,z) : 0 \leq z \leq 9-x-2y,\ x^2+y^2 \leq 4\}$ if the mass-density of E at (x,y,z) is proportional to the distance from (x,y,z) to the z-axis.
5. Compute the mass of the solid $E=\{(x,y,z) : 0 \leq z \leq 12e^{-(x^2+y^2)},\ x^2+y^2 \leq 4,\ x \geq 0,\ y > 0\}$ if E has constant mass density.
6. Let E be the solid region bounded by the cylinder $x^2+y^2=1$, the paraboloid $z=x^2+y^2$ and the plane $z=0$. Compute $\iiint_E x^2 z\ dV$.
7. Evaluate the triple integral by converting to cylindrical coordinates.
   (a) $\iiint_E (x^2y+y^3)\ dV$, where E is the solid beneath $z=4-x^2-y^2$ in the first octant.
   (b) $\iiint_E y\ dV$, where E is bounded by the planes $z=0$, $z=x+y+10$ and the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.
   (c) $\iiint_E yz\ dV$, where E is the part of the solid beneath the paraboloid $z=9-x^2-y^2$ in the first octant.
8. Compute the volume of the region bounded by the plane $z=0$, the cylinder $x^2+y^2=2x$ and the cone $z=\sqrt{x^2+y^2}$.

9. For each given solid E, find the region E' whose image under the rectangular-to-spherical coordinate transformation (equations (3.6)) is E.

   (a) E is the solid in the first octant between the spheres $x^2+y^2+z^2=16$ and $x^2+y^2+z^2=9$ and the planes $y=0$, $y=x$.

   (b) $E=\{(x,y): z^2 \leq 3(x^2+y^2),\ x^2+y^2+z^2 \leq 25\}$.

10. Evaluate the triple integral by converting to spherical coordinates.

    (a) $\iiint_E (x^2+y^2+z^2)^3\, dV$, where $E=\{(x,y,z): x^2+y^2+z^2 \leq 16\}$.

    (b) $\iiint_E y^2\, dV$, where E is bounded by the hemispheres $x=\sqrt{1-y^2-z^2}$, $x=\sqrt{16-y^2-z^2}$ and the plane $x=0$.

    (c) $\iiint_E z\, dV$, where E is the llid in the first octant bounded by the planes $y=0$, $y-\sqrt{3}x$, the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 \leq 9$.

    (d) $\iiint_E \sqrt{x^2+y^2+z^2}\, dV$, where E is bounded by the sphere $x^2+y^2+z^2=z$

11. Sketch the solid whose volume is given by the iterated integral. Convert the iterated integral to an integral in cylindrical coordinates, then evaluate the integral.

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_2^{3/\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

12. Evaluate the iterated integral by converting to cylindrical coordinates.

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} z\, dz\, dy\, dx.$$

3.7 Change of Variables in Triple Integrals

Definition 3.7.1. Let E' be an open subset of $\mathbb{R}^3$ and suppose $T: E' \to E$ is a one-to-one transformation given by

$$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w)).$$

Suppose each of T's component functions have continuous first-order partial derivatives in E'. The Jacobian $J_T(u,v,w)$ of T is

$$J_T(u,v,w) = \det \begin{pmatrix} \dfrac{\partial x}{\partial u}(u,v,w) & \dfrac{\partial x}{\partial v}(u,v,w) & \dfrac{\partial x}{\partial w}(u,v,w) \\ \dfrac{\partial y}{\partial u}(u,v,w) & \dfrac{\partial y}{\partial v}(u,v,w) & \dfrac{\partial y}{\partial w}(u,v,w) \\ \dfrac{\partial z}{\partial u}(u,v,w) & \dfrac{\partial z}{\partial v}(u,v,w) & \dfrac{\partial z}{\partial w}(u,v,w) \end{pmatrix}$$

Theorem 3.7.2 (Change of Variables inTriple Integrals). *Suppose $T : E' \to E$ is given by*

$$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w)) .$$

where each of T's component functions x, y and z have continuous first-order partial derivatives on E'. Suppose $J_T(u,v,w) \neq 0$ except possibly on $\partial E'$. If $f(x,y,z)$ is continuous on E and E is bounded by piecewise smooth surfaces, then

$$\iiint_E f(x,y,z)\, dV = \iiint_{E'} f(x(u,v,w), y(u,v,w), z(u,v,w))\ |J_T(u,v,w)|\ dV'.$$

Example 3.7.3. Use a triple integral to compute the volume enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution

Example 3.7.4. Let a, b and c be positive constants. Compute the volume of the solid bounded by

$$\left(\frac{x}{a}\right)^{1/3} + \left(\frac{y}{b}\right)^{1/3} + (zc)^{1/3} = 1$$

with $x \geq 0$, $y \geq 0$ and $z \geq 0$.

Solution

3.7.1 Exercises

1. Let E be the part of the solid bounded by $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$ that lies in the first octant. Use a triple integral and the change of variables $u=\sqrt{x}$, $v=\sqrt{y}$, $w=\sqrt{z}$ to compute the volume of E.

2. Let $a,b,c>0$ be constants. Use a triple integral with a suitable change of variables to compute the volume of the solid bounded by the given surface.

   (a)
   $$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1.$$

   (b)
   $$\left(\frac{x}{a}\right)^{1/3}+\left(\frac{y}{b}\right)^{1/3}+\left(\frac{z}{c}\right)^{1/3}=1,$$

   where $x\geq 0$, $y\geq 0$ and $z\geq 0$.

3. Evaluate $\iiint_E z\,dV$, where E is bounded below by the cone $z=\sqrt{x^2/4+y^2/9}$ and above by the ellipsoid $x^2/4+y^2/9+z^2=1$.

4. Evaluate $\iiint_E (4x^2-9y^2)\ dV$, where E is enclosed by the paraboloid $z=x^2/9+y^2/4$ and the plane $z=10$.

5. Use a suitable change of variables to evaluate $\iiint_E f(x,y,z)\,dV$, where $f(x,y,z)=1$ and E is bounded by the planes $y-2x=0$, $y-2x=1$, $z-3y=0$, $z-3y=1$, $z-4x=0$ and $z-4x=3$.

3.8 Exam 3 Review

Problem 3.1. Consider the iterated integral

$$\int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx.$$

1. Sketch the region whose area is represented by the iterated integral.

2. Evaluate the integral.

3. Change the order of integration then evaluate the integral.

Problem 3.2. Let D be the region bounded by the curves

$$x = y^2, \quad 4x = y^2, \quad xy = 1 \quad \text{and} \quad xy = 2.$$

Use a double integral with a suitable change of variables to compute the area of D.

Problem 3.3. Let D be the region bounded by the curves

$$x^2 + y^2 = 2x, \quad x^2 + y^2 = 4x, \quad y = -x, \quad \text{and} \quad y = x.$$

Suppose the mass-density of D at (x, y) is inversely proportional to the distance from (x, y) to the y-axis. Compute the center of mass of D.

Problem 3.4. Consider the iterated integral

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx,$$

where a is a positive constant.

1. Sketch the solid whose volume is represented by the iterated integral.
2. Evaluate the iterated integral. Check your answer by comparing with a suitable geometric formula.

Problem 3.5. Let E be the volume of the solid that lies in octant one and is bounded by the spheres $x^2+y^2+z^2 = a^2$, $x^2+y^2+z^2 = b^2$, the cones $z = \sqrt{x^2 + y^2}$, $z = \sqrt{3(x^2 + y^2)}$ and the planes $y = 0$, $y = x$. Use a triple integral to compute the volume of E.

Problem 3.6. Let E be the solid bounded by

$$z = x^2 + y^2, \qquad y = x^2, \qquad y = 1 \qquad \text{and} \qquad z = 0.$$

1. Sketch both E and the projection of E onto the xy-plane.
2. Suppose $f(x, y, z)$ is a function that is integrable on E. Express $\iiint_E f(x, y, z)\, dV$ as an iterated integral (you will not be able to evaluate because I didn't give you f).

Problem 3.7. Let E be the solid bounded by

$$x^2 + y^2 = a^2, \qquad y = z, \qquad \text{and} \qquad z = 0$$

corresponding to $z \geq 0$.

1. Sketch E.

2. Evaluate $\iiint_E y \, dV$.

Problem 3.8. Let D be the planar region which lies inside the curve

$$(x^2+y^2)^2 = 4(x^2-y^2),$$

and outside the curve $x^2+y^2=2$ corresponding to $x \geq 0$.

1. Sketch D.

2. Use a double integral to compute the area of D.

Problem 3.9. Let $E = \{(x,y,z) : x^2+y^2+z^2 \leq 2z\}$.

1. Sketch E.

2. If E has uniform mass-density, compute the moment of inertia of E about the z-axis.

3.9 Line Integrals

3.9.1 Frequently Used Parameterizations of Curves

1. Ellipse with center (a,b), x-radius R_1 and y-radius R_2 in the xy-plane:

$$\mathbf{r}(t) = \langle a + R_1 \cos t, b + R_2 \sin t \rangle, \qquad 0 \leq t \leq 2\pi$$

   Special case: If the x and y-radii coincide, this parameterizes a circle.

2. Line segment with initial point (x_0,y_0,z_0) and final point (x_1,y_1,z_1).

$$\begin{aligned}\mathbf{r}(t) &= (1-t)\langle x_0,y_0,z_0\rangle + t\langle x_1,y_1,z_1\rangle \\ &= \langle (1-t)x_0+tx_1,\ (1-t)y_0+ty_1,\ (1-t)z_0+tz_1\rangle, \qquad 0 \leq t \leq 1\end{aligned}$$

   Similar in one and two dimensions.

3. One turn of a helix whose projection onto the xy plane is a circle of radius R and center $(0, 0, 0)$ and whose z-growth is linear

$$\mathbf{r}(t) = \langle R\cos t, R\sin t, t\rangle \qquad 0 \leq t \leq 2\pi.$$

This parameterization traverses the helix counterclockwise as viewed from above.

4. Graph of $y = f(x)$ (contained in xy-plane) corresponding to $a \leq x \leq b$

$$\mathbf{r}(t) = \langle t, f(t)\rangle\,, \qquad a \leq t \leq b.$$

3.9.2 Type-I Line Integrals

Suppose C is a curve in space (or in the xy-plane) with smooth parameterization $\mathbf{r}(t)$ for $a \leq t \leq b$. Recall the arclength function $s(t)$ from equation (1.18)

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\ du, \qquad \text{for } a \leq t \leq b.$$

By the fundamental theorem of calculus, $ds/dt = \|\mathbf{r}'(t)\|$, or $ds = \|\mathbf{r}'(t)\|\ dt$.

Definition 3.9.1. 1. Suppose C_1 is a curve with initial point (x_0, y_0, z_0) and final point (x_1, y_1, z_1) and suppose C_2 is a curve with initial point (x_1, y_1, z_1) and final point (x_2, y_2, z_2). The curve $C_1 + C_2$ is defined to be the curve with initial point (x_0, y_0, z_0) and final point (x_2, y_2, z_2) that traces C_1 first then C_2.

2. The curve that traces C_1 from its final point (x_1, y_1, z_1) to its initial point (x_0, y_0, z_0) is denoted $-C_1$.

Definition 3.9.2. Let C be a curve of finite arclength and let f be a bounded $\mathbb{R}$-valued function on C. Partition C into N pairwise adjacent subcurves C_i of length Δs_i, where $i = 1, 2, \cdots, N$. Let $\mathbf{r}_i^* \in C_i$ and suppose $\Delta s_i \to 0$ as $N \to \infty$. The (type-I) line integral of f over C is

$$\int_C f(\mathbf{r})\, ds = \lim_{N\to\infty} \sum_{i=1}^{N} f(\mathbf{r}_i^*)\Delta s_i$$

provided the limit exists.

Proposition 3.9.3 (Properties of Type-I Line Integrals). *Let C be a piecewise smooth curve. Suppose $f(\mathbf{r})$ and $g(\mathbf{r})$ are integrable functions on C and that a is a constant. The following properties hold*

1. *Linearity:*

$$\int_C (af(\mathbf{r}) + g(\mathbf{r}))\, ds = a\int_C f(\mathbf{r})\, ds + \int_C g(\mathbf{r})\, ds.$$

2. *Additivity: If $C = C_1 + C_2$, then*

$$\int_{C_1+C_2} f(\mathbf{r})\, ds = \int_{C_1} f(\mathbf{r})\, ds + \int_{C_2} f(\mathbf{r})\, ds.$$

3. *Monotonicity: If $f(\mathbf{r}) \le g(\mathbf{r})$ for all $\mathbf{r} \in C$, then*

$$\int_C f(\mathbf{r})\, ds \le \int_C g(\mathbf{r})\, ds.$$

Proposition 3.9.4 (Evaluating Line Integrals). *Suppose f is a continuous function whose domain is a curve C and suppose C has a smooth parameterization $\mathbf{r}(t)$ for $a \leq t \leq b$. Then*

$$\int_C f(\mathbf{r})\ ds = \int_a^b f(\mathbf{r}(t))\ \|\mathbf{r}'(t)\|\ dt \tag{3.7}$$

Here is a list of steps for evaluating $\int_C f(\mathbf{r})\ ds$.

1. Find a parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for C such that C is traversed only once for $a \leq t \leq b$.
2. Compute both $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.
3. Substitute $x = x(t)$, $y = y(t)$ and $z = z(t)$ in f's formula.
4. Evaluate $\int_C f(\mathbf{r})\ ds$ by evaluating the right-hand side of equation (3.7) (which is an ordinary calc I integral).

Remark 3.9.5.

1. $\int_C f(\mathbf{r})\ ds$ is independent of the choice of parameterization for C.
2. If $f = 1$, then $\int_C f(\mathbf{r})\ ds$ gives the length of C.

Example 3.9.6. Evaluate the line integral of $f(x, y) = xy^2$ over the portion of the curve $x^2 + y^2 = 9$ that corresponds to $x \geq 0$.

Solution

1. Parameterize C:

2. Compute $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$:

3. Substitute x and y according to the formulas

$$\begin{cases} x = \\ y = \end{cases}$$

4. Evaluate the right-hand side of (3.7):

Example 3.9.7. Let C be the curve determined by the portion of the intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = 9$ that lies in octant one. Compute $\int_C z \, ds$.

Solution

1. Parameterize C:

2. Compute $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$:

3. Substitute x, y and z according to the formulas

$$\begin{cases} x = \\ y = \\ z = \end{cases}$$

4. Evaluate the right-hand side of (3.7):

Definition 3.9.8. Let C be a smooth curve in xyz-space with linear mass-density $\sigma(x, y, z)$.

1. The mass of C is

$$m = \int_C \sigma(x, y, z)\, ds.$$

2. The center of mass of C is the ordered triple $(\overline{x}, \overline{y}, \overline{z})$, where

$$\begin{aligned}
\overline{x} &= \frac{1}{m}\int_C x\sigma(x, y, z)\, ds \\
\overline{y} &= \frac{1}{m}\int_C y\sigma(x, y, z)\, ds \\
\overline{z} &= \frac{1}{m}\int_C z\sigma(x, y, z)\, ds
\end{aligned}$$

3. The moments of inertia of C about the x, y and z axes are

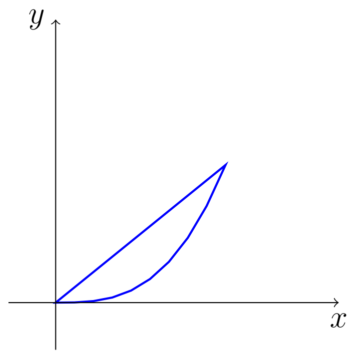
$$\begin{aligned}
I_x &= \int_C (y^2 + z^2)\sigma(x, y, z)\, ds \\
I_y &= \int_C (x^2 + z^2)\sigma(x, y, z)\, ds \\
I_z &= \int_C (x^2 + y^2)\sigma(x, y, z)\, ds
\end{aligned}$$

respectively.

Example 3.9.9. Let C be the part of the graph of the function $y = e^x$ corresponding to $0 \leq x \leq 4$. Suppose the linear mass-density of C at (x, y) is proportional to the distance from (x, y) to the x-axis. Compute the center of mass of C.

Solution

Example 3.9.10. Let C be the closed curve generated by $y = x^2$ and $y = x$ as pictured below



Compute $\int_C x\ ds$.

Solution

3.9.3 Exercises

1. Evaluate $\int_C (x^2 - y + 3z)\, ds$ if C is the segment with initial point $(-1, -2, -1)$ and final point $(0, 0, 0)$.

2. Evaluate $\int_C f(x, y)\, ds$.

   (a) $f(x, y) = x + y$, C is the part of the curve $x^2 + y^2 = 1$ in quadrant one.

   (b) $f(x, y) = y + 2$, C is the graph of the function $1 - x^2$ for $0 \leq x \leq 1$.

   (c) $f(x, y) = x^2 - y^2 + 4$, C is the portion of the curve $x^2 + y^2 = 4$ that lies in quadrant one.

   (d) $f(x, y) = x \sin y$, C is the segment from with endpoints $(0, 1)$ and $(5, 0)$.

   (e) $f(x, y) = x + y$, C is the boundary of the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.

   (f) $f(x, y) = xy$, C is the portion of the ellipse $(x/a)^2 + (y/b)^2 = 1$.

3. Evaluate $\int_C f(x, y, z)\, ds$.

   (a) $f(x, y, z) = xyz$, C is the helix parameterized by $x = 2\cos t$, $y = -2\sin t$, $z = t$ for $0 \leq t \leq \pi/2$.

   (b) $f(x, y, z) = 3x + z$, C is the curve parameterized by $x = t$, $y = t^2$, $z = t^3$ for $1 \leq t \leq 2$.

   (c) $f(x, y, z) = 3x^2 + 3y^2 - z^2$, C is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 0$

   (d) $f(x, y, z) = x^2 + y^2 + z^2$, C is the helix parameterized by $x(t) = a\cos t$, $y(t) = a\sin t$, $z(t) = bt$ for $0 \leq t \leq 2\pi$.

4. Let C be the spring parameterized by $\mathbf{r}(t) = \langle \cos t \sin t, t \rangle$ for $\pi \leq t \leq 4\pi$. Suppose the linear mass-density of C at (x, y, z) is proportional to the distance from (x, y, z) to the xy-plane. Compute the mass of C.

5. Compute $\int_C \sqrt{x^2 + y^2}\, ds$ if C is the circle $x^2 + y^2 = ax$.

6. Let C be a semicircular rod of radius R with uniform linear mass-density.

   (a) Compute the mass of C.

   (b) Compute the center of mass of C.

   (c) Compute the moment of inertia of C about the axis containing the endpoints of C.

3.10 Surface Integrals

3.10.1 Frequently Used Parameterizations

1. Sphere of radius a and center $(0,0,0)$

$$\mathbf{r}(u,v) = \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle, \qquad 0 \le u \le \pi, 0 \le v \le 2\pi.$$

2. Graph of a function $g(x,y)$ with domain D

$$\mathbf{r}(x,y) = \langle x, y, g(x,y)\rangle, \qquad (x,y) \in D.$$

3.10.2 Parameterizing Surfaces

Definition 3.10.1. A parameterization of a surface S in $\mathbb{R}^3$ is a function

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, \qquad (u,v) \in D,$$

where D is a two-dimensional domain called the parameter domain. We say $\mathbf{r}$ a smooth parameterization if the partial derivatives $\mathbf{r}_u$, $\mathbf{r}_v$ are continuous and if $\|\mathbf{r}_u \times \mathbf{r}_v\| \neq 0$.

Example 3.10.2. Find a parameterization for the part of a sphere of radius R and center $(0,0,0)$ that lies above the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$ in two ways; viewing the surface as the graph of a function, and viewing the surface as a piece of a sphere.

Solution

3.10.3 Surface Integrals

Definition 3.10.3. Suppose f is a bounded $\mathbb{R}$-valued function defined on a surface S. Let $\{S_p : p = 1, \cdots, N\}$ be a partition of S into subsurfaces S_p whose surface areas are ΔS_p and let $\mathbf{r}_p^* \in S_p$. Suppose $\Delta S_p \to 0$ as $N \to \infty$. The surface integral of f over S is

$$\int\int_S f(\mathbf{r})\, dS = \lim_{N\to\infty} \sum_{p=1}^{N} f(\mathbf{r}_p^*)\Delta S_p$$

provided the limit exists.

Proposition 3.10.4 (Properties of Surface Integrals). *Suppose f and g are integrable on a surface S and suppose c is a constant.*

1. *Linearity:*

$$\int\int_S (f(x,y,z) + cg(x,y,z))\, dS = \int\int_S f(x,y,z)\, dS + c\int\int_S g(x,y,z)\, dS.$$

2. *Additivity: If S is cut into two disjoint subsurfaces S_1 and S_2, then*

$$\int\int_S f(x,y,z)\, dS = \int\int_{S_1} f(x,y,z)\, dS + \int\int_{S_2} f(x,y,z)\, dS$$

3. *Monoticity: If $f(x,y,z) \le g(x,y,z)$, then*

$$\int\int_S f(x,y,z)\, dS \le \int\int_S g(x,y,z)\, dS$$

Proposition 3.10.5 (Evaluating Surface Integrals). *Let S be a surface with smooth parameterization*

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, \qquad (u,v) \in D.$$

Suppose f is a 'nice enough' $\mathbb{R}$-valued function on S. Then

$$\int\int_S f(\mathbf{r})\, dS = \int\int_D f(\mathbf{r}(u,v))\, \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA. \tag{3.8}$$

Remark 3.10.6. Follow these steps to evaluate $\int\int_S f(\mathbf{r})\,dS$

1. Find a smooth parameterization
$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle\,, \qquad (u,v)\in D$$
for S. This step includes finding the parameter domain D. Your parameterization should cover S exactly once as (u,v) ranges through D.
2. Compute $\|\mathbf{r}_u \times \mathbf{r}_v\|$.
3. Substitute $x = x(u,v)$, $y = y(u,v)$ and $z = z(u,v)$ into the integrand.
4. Write the right-hand side of equation (3.8) as in iterated integral in a suitable coordinate system and evaluate.

Example 3.10.7. Let S be the part of the cone $z = \sqrt{x^2+y^2}$ corresponding to $0 \le z \le 4$. Compute $\iint_S (8-z)\,dS$.

Solution

Example 3.10.8. Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$. Compute $\int\int_S z\, dS$. What changes if S is chosen to be the part of the sphere below $z = 1$?

Solution

Definition 3.10.9. Let S be a smooth surface. The surface area $A(S)$ of S is

$$A(S) = \iint_S dS.$$

Example 3.10.10. Let S be the part of the plane $z = 12 - 4x - 3y$ that lies inside the cylinder $x^2/4 + y^2 = 1$. Compute the area of S.

Solution

Definition 3.10.11. Let S be a smooth surface and let $\sigma(x, y, z)$ be a mass-density function on S.

1. The mass m of S is

$$m = \int\int_S \sigma(x, y, z)\, dS.$$

2. The center of mass of S is the ordered triple $(\overline{x}, \overline{y}, \overline{z})$, where

$$\begin{aligned}
\overline{x} &= \frac{1}{m}\int\int_S x\sigma(x, y, z)\, dS \\
\overline{y} &= \frac{1}{m}\int\int_S y\sigma(x, y, z)\, dS \\
\overline{z} &= \frac{1}{m}\int\int_S z\sigma(x, y, z)\, dS.
\end{aligned}$$

3. The moments of inertia about the x, y and z axes of S are

$$\begin{aligned}
I_x &= \int\int_S (y^2 + z^2)\, \sigma(x, y, z)\, dS \\
I_y &= \int\int_S (x^2 + z^2)\, \sigma(x, y, z)\, dS \\
I_z &= \int\int_S (x^2 + y^2)\, \sigma(x, y, z)\, dS
\end{aligned}$$

respectively.

Example 3.10.12. Find the center of mass of the cone of height h and base radius a if the mass density of the cone is uniform.

Solution

3.10.4 Exercises

1. Find a parameterization for the given surface S.

   (a) S is the part of the elliptic cylinder $x^2/a^2 + y^2/b^2 = 1$ the lies between the planes $z = -1$ and $z = 1$.

   (b) S is the part of the circular cylinder that lies between the planes $z = 0$ and $z - y = 1$.

   (c) S is the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

   (d) S is the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies below the cone $z = \sqrt{x^2 + y^2}$.

2. Compute the surface area of the surface S.

   (a) S is the part of the plane in octant one that intersects the x, y and z axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

   (b) S is the part of the plane $2x + 3y + z = 2$ that lies inside the cylinder $x^2 + y^2 = 4$.

   (c) S is the part of the paraboloid $z = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 16$.

   (d) S is the part of the sphere $x^2 + y^2 + z^2 = 25$ cut out by the cylinder $x^2 + y^2 = 25y$.

3. Evaluate $\int\int_S f(x, y, z)\, dS$.

   (a) S is the part of the plane $3x + 2y + z = 1$ that lies in octant one; $f(x, y, z) = yz$.

   (b) S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 4$; $f(x, y, z) = x^2y^2$.

   (c) S is the boundary of the solid $\sqrt{x^2 + y^2} \leq z \leq 1$; $f(x, y, z) = x^2 + y^2$. [*Hint: Use additivity of surface integral.*]

   (d) S is the part of the paraboloid $z = x^2 + y^2$ in octant one the lies below $z = 1$; $f(x, y, z) = xyz$.

4. Find the center of mass of a hemisphere of radius R whose mass-density is uniform.

5. Compute the mass of the surface of bounded by $z = \frac{1}{\sqrt{3}}\sqrt{x^2 + y^2}$ and the plane $z = 2/\sqrt{3}$ if the mass-density at (x, y, z) is proportional to the square distance from (x, y, z) to the origin.

6. Compute the mass of the surface S defined by $f(x, y) = 4 - 2\sqrt{x^2 + y^2}$ for $0 \le z \le 4$ if the mass-density at (x, y, z) is proportional to the distance from (x, y, z) to the z-axis.

7. Compute the moment of inertia of a sphere of radius R and uniform mass-density about a diameter of the sphere.

Chapter 4

Vector Calculus

4.1 Vector Fields

Definition 4.1.1.

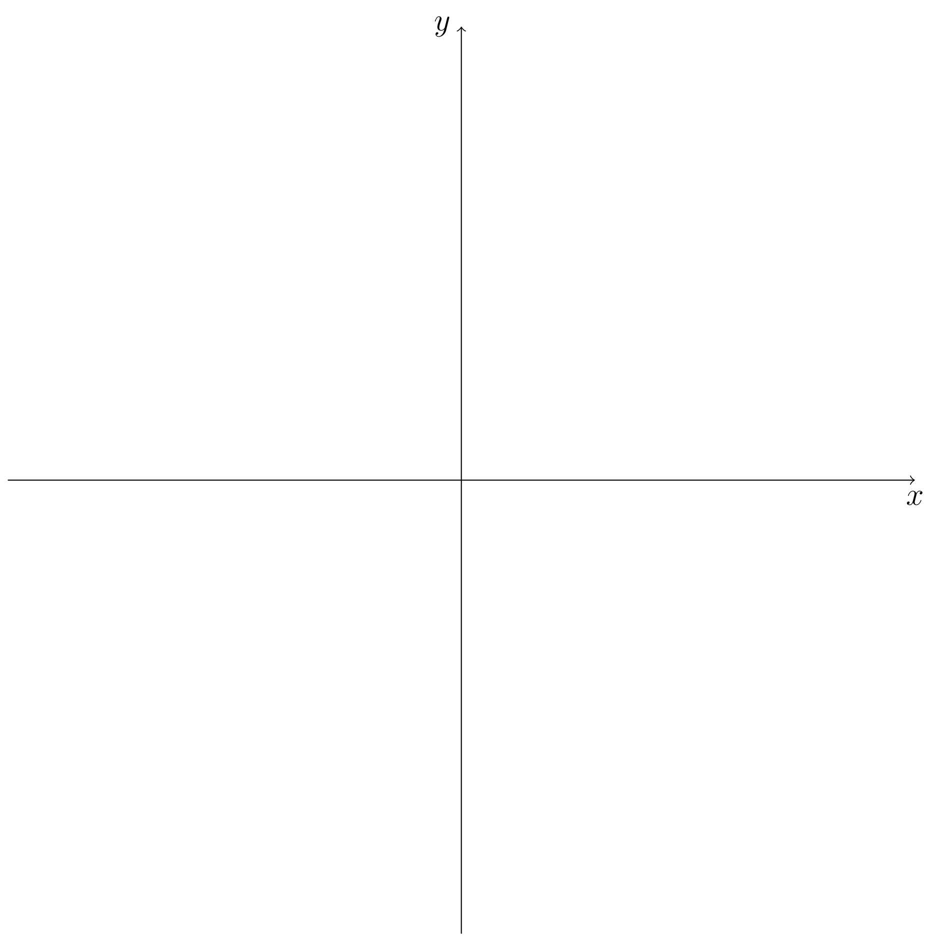
1. Let D be a planar region. A vector field on D is a function $\mathbf{F}(x,y) = \langle F_1(x,y), F_2(x,y)\rangle$ for $(x,y) \in D$ (i.e. $\mathbf{F}$ assigns each point (x,y) of D the two-vector $\langle F_1(x,y), F_2(x,y)\rangle$).

2. Let E be a solid region. A vector field on E is a function $\mathbf{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle$ for $(x,y,z) \in E$ (i.e. $\mathbf{F}$ assigns each point (x,y,z) of E the three-vector $\langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle$).

Definition 4.1.2. If f is a real-valued function whose first order partials exist then ∇f, the gradient of f, is a vector field called the gradient field of f.

Example 4.1.3. Let $f(x,y) = 1/(x^2+y^2)$ for $(x,y) \neq (0,0)$. Plot the Gradient field of f.

Solution



4.1.1 Exercises

1. Sketch the vector field $\mathbf{F}(x, y) = \langle -y, x \rangle / \sqrt{x^2 + y^2}$ for $(x, y) \neq (0, 0)$.
2. Sketch the gradient field for $f(x, y) = \arctan(y/x)$.

4.2 Type-II Line Integrals

Motivation: The work W done by a constant force $\mathbf{F}$ in moving a point object from $\mathbf{r}_1$ to $\mathbf{r}_2$ is

$$W = \mathbf{F} \cdot \Delta\mathbf{r},$$

where $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.
Now suppose $\mathbf{F}$ were non constant (but continuous) and a point object is moved by $\mathbf{F}$ from an initial point $\mathbf{r}_1$ to a final point $\mathbf{r}_2$ along a curve C. The work done can be approximated as indicated in the following picture:

$$W \approx \mathbf{F}(\mathbf{r}_1^*) \cdot \Delta\mathbf{r} + \mathbf{F}(\mathbf{r}_2^*) \cdot \Delta\mathbf{r} + \cdots + \mathbf{F}(\mathbf{r}_m^*) \cdot \Delta\mathbf{r}.$$

In the limiting case that the curve is cut into infinitely small pieces we get

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Definition 4.2.1 (Type II Line Integral). Let $\mathbf{F}$ be a vector field defined on a smooth curve C. The line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \widehat{\mathbf{T}}\, ds,$$

where $\widehat{\mathbf{T}}$ is the unit tangent vector to C and s is the arclength parameter.

Definition 4.2.2.

1. Suppose C is a curve with initial point $\mathbf{r}_1$ and final point $\mathbf{r}_2$. The curve $-C$ is the curve that starts at $\mathbf{r}_2$ and ends at $\mathbf{r}_1$ and traverses C in the opposite direction.

2. If C_1 and C_2 are curves such that the initial point of C_2 and the final point of C_1 coincide, then $C_1 + C_2$ is the curve whose initial point is the initial point of C_1 and whose final point is the final point of C_2 and that traces C_1 followed by C_2.

Remark 4.2.3. Suppose C_1 and C_2 are curves with smooth parameterizations

$$\begin{aligned} &\mathbf{r}(t) && a \le t \le b \\ &\mathbf{s}(t) && b \le t \le c \end{aligned}$$

respectively. Then $\mathbf{r}(-t)$, $-b \le t \le -a$ is a parameterization for $-C_1$ and

$$\mathbf{R}(t) = \begin{cases} \mathbf{r}(t) & \text{if } a \le t \le b \\ \mathbf{s}(t) & \text{if } b \le t \le c \end{cases}$$

is a parameterization for $C_1 + C_2$.

Proposition 4.2.4 (Properties of Type-II Line Integrals). *Suppose* $\mathbf{F}$ *is an integrable vector field and* C*,* C_1 *and* C_2 *are smooth curves. Then*

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

and

$$\int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Proposition 4.2.5 (Evaluating Line Integrals). *Let* $\mathbf{F}$ *be a continuous vector field defined on a curve* C *and suppose* C *has smooth parameterization* $\mathbf{r}(t)$*, where* $a \le t \le b$*. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \tag{4.1}$$

Remark 4.2.6. Unlike Type-I line integrals, Type-II line integrals are not independent of the chosen parameterization. Specifically, Type-II line integrals are only independent of change of parameterization within an orientation class.

Remark 4.2.7 (Evaluating Type-II line integrals). Follow these steps to evaluate the Type-II line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$

1. Find a piecewise-smooth parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$ of C (finding the parameter domain $[a, b]$ is part of this step). Your parameterization should only cover C once. Keep in mind that orientation of C matters.
2. Compute $\mathbf{r}'(t)$.
3. Substitute $x = x(t)$, $y = y(t)$ and $z = z(t)$ respectively as determined by $\mathbf{r}(t)$.
4. Compute $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$.
5. Evaluate the right-hand side of equation (4.1) (which is an ordinary calc-I integral).

Example 4.2.8.

1. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F}(x, y, z) = \langle z, xy, yz \rangle$ and C is the upper half of the circle $x^2 + y^2 = 1$ oriented counterclockwise.
2. Repeat with C replaced by $-C$.

Solution

Remark 4.2.9. Suppose $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is a vector field and C is a curve with smooth parameterization $\mathbf{r}(t) = \langle x, y, z \rangle$. Then

$$d\mathbf{r} = \langle dx, dy, dz \rangle \, .$$

Consequently, an alternative and frequently used notation for $\mathbf{F} \cdot d\mathbf{r}$ is the the following.

$$\mathbf{F} \cdot d\mathbf{r} = \langle F_1, F_2, F_3 \rangle \cdot \langle dx, dy, dz \rangle = F_1 \; dx + F_2 \; dy + F_3 \; dz.$$

Example 4.2.10. Evaluate $\int_C y \; dx + x \; dy$, where C is the closed contour consisting of the segment starting at $(0,0)$ and ending at $(5,5)$ followed by the circular arc starting at $(5,5)$ and ending at $(0, 5\sqrt{2})$, then followed by the segment starting at $(0, 5\sqrt{2})$ and ending at $(0,0)$.

Solution

Example 4.2.11. Let C be the closed contour consisting of the following two parts

1. A single turn of a helix of radius 2 that winds about the z-axis counterclockwise as viewed from above. The initial point is $(2, 0, 0)$ and the terminal point is $(2, 0, 4)$. The z-growth is linear.
2. The straight line segment with initial point $(2, 0, 4)$ and final point $(2, 0, 0)$.

Compute the work done by the force $\mathbf{F}(x, y, z) = \langle -y, x, z^2 \rangle$ in moving a point object along C.

Solution

4.2.1 Exercises

1. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

   (a) $\mathbf{F} = \langle 0, y, xy \rangle$. C is the curve parameterized by $\mathbf{r}(t) = \langle 0, t^2, t^3 \rangle$ for $0 \leq t \leq 1$.

   (b) $\mathbf{F} = \langle -y, x, z \rangle$. C is the boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ that lies in octant one. C is oriented counterclockwise as viewed from above.

   (c) $\mathbf{F} = \langle -z, 0, x \rangle$. C is the boundary of the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies in octant one. C is oriented clockwise as viewed from above.

   (d) $\mathbf{F} = \langle y, -xz, y(x^2 + y^2) \rangle$. C is the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $x + y + z = 1$ oriented counterclockwise as viewed from the top of the y-axis.

   (e) $\mathbf{F} = \left\langle e^x e^{\sqrt{y}}, 0 \right\rangle$. C is the part of the parabola $x = y^2$ in the xy-plane with initial point $(0, 0)$ and final point $(1, 1)$.

   (f) $\mathbf{F} = \langle z^{-1}, y^{-1}, x^{-1} \rangle$. C is the line segment with initial point $(1, 1, 1)$ and final point $(4, 2, 8)$.

2. Compute the work done by the force $\mathbf{F}$ in moving a point object along the curve C.

   (a) $\mathbf{F} = \langle -y, x, 1 \rangle$. C is the circle $x^2 + y^2 = 1$ with $z = 0$ oriented clockwise as viewed from above.

   (b) $\mathbf{F} = \langle x^2, -xy \rangle$. C is the curve $x = \cos^3 t$, $y = \sin^3 t$ with initial point $(1, 0)$ and final point $(0, 1)$.

   (c) $\mathbf{F} = \langle -y, -x \rangle$. C is the counterclockwise-oriented circular arc $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(0, 2)$.

4.2.2 Fundamental Theorem of Line Integrals

Definition 4.2.12. A vector field $\mathbf{F}$ in a region E is called conservative if there exists a scalar-valued function f, called a potential function, such that $\nabla f = \mathbf{F}$.

Example 4.2.13. Show that

$$\mathbf{F}(x, y, z) = \left\langle z + \frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, x \right\rangle$$

is conservative on $\mathbb{R}^3$ by finding a potential function $f(x, y, z)$.

Solution

Theorem 4.2.14 (Fundamental Theorem of Line Integrals). *Let C be a smooth curve in a region E with initial point $\mathbf{r}_1$ and final point $\mathbf{r}_2$. If $\mathbf{F}$ is a continuous conservative vector field in E with potential function f, then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}_2) - f(\mathbf{r}_1).$$

Example 4.2.15. Let $\mathbf{F}(x, y, z) = \langle yz, xz + 2y, xy + y + 2z\rangle$. let C be the curve with initial point $A = (0, 0, 0)$ and final point $B = (1, 1, 1)$ that consists of the segments $\overline{AB_1}$, $\overline{B_1B_2}$ and $\overline{B_2B}$, where $B_1 = (1, 9, 5)$ and $B_2 = (3, 3, 8)$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Solution

Definition 4.2.16. Let $\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle$ be a differentiable vector field. The curl of $\mathbf{F}$ is

$$\operatorname{curl}\mathbf{F} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle.$$

Remark 4.2.17. An easy way to remember the formula for $\operatorname{curl}\mathbf{F}$ is via the formal determinant

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}.$$

Example 4.2.18. Compute the curl of the vector field $\mathbf{F} = \langle yz, xyz, x^2\rangle$.

Solution

Theorem 4.2.19. *If* $\mathbf{F}$ *is a conservative vector field in* E *and the component functions of* $\mathbf{F}$ *have continuous first-order partials in* E*, then* $\operatorname{curl}\mathbf{F} = \mathbf{0}$ *in* E*.*

Definition 4.2.20. Let C be a curve.

1. C is *simple* if C has no points of self intersection except possibly the endpoints of C.

2. C is *closed* if the initial point and the final point of C coincide.

3. C is a *loop* if the C is both simple and closed.

Definition 4.2.21 (Connected/ Simply Connected). 1. An open set E is called connected if for all pairs of points $\mathbf{r}_1$ and $\mathbf{r}_2$ in E, there is a path connecting $\mathbf{r}_1$ and $\mathbf{r}_2$ that lies entirely in E.

2. A connected open set E is called simply connected if every simple closed curve that lies entirely in E can be deformed (without tearing or breaking) to a single point while staying entirely in E.

Remark 4.2.22. Roughly speaking, if E is a two dimensional open connected region, then E is simply connected if E has no 'holes'. The situation is more subtle if E is three dimensional.

Example 4.2.23. Which of the following regions are simply connected?

$$\begin{aligned} E_1 &= \left\{(x,y,z) : \frac{1}{4} \leq x^2+y^2+z^2 \leq 1\right\} \\ E_2 &= \{(x,y,z) : 0 < x^2+y^2 \leq 1\} \\ E_3 &= \{(x,y,z) : x^2+y^2 \leq 1\} \end{aligned}$$

Solution

Theorem 4.2.24. *Suppose* $\mathbf{F}$ *is a vector field with continuous first-order partials on a simply connected region* E*. If* $\operatorname{curl} \mathbf{F} = \mathbf{0}$*, then* $\mathbf{F}$ *is conservative on* E*.*

Theorem 4.2.25 (Independence of Path). *Suppose* $\mathbf{F}$ *is conservative on a simply connected region* E *and that* C_1 *and* C_2 *are smooth curves in* E *whose initial points coincide and whose terminal points coincide. Then*

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Corollary 4.2.26. *If* $\mathbf{F}$ *is a continuous conservative vector field on a simply connected region* E *then* $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for every piecewise-smooth simple closed curve* C *that lies entirely in* E*.*

Example 4.2.27. Redo example 4.2.15 using the path-independence property.

Solution

Example 4.2.28. Let

$$\mathbf{F}(x, y) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

1. Show that $f(x, y) = \arctan(y/x)$ is a potential function for $\mathbf{F}$.
2. Let C be the curve $x^2 + y^2 = 1$ oriented counterclockwise. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.
3. Does this contradict the path-independence property (Theorem 4.2.25 or Corollary 4.2.26)?

Solution

4.2.3 Exercises

1. Compute the curl of the vector field.
   (a) $\mathbf{F} = \langle xyz, 0, -xy^2 \rangle$
   (b) $\mathbf{F} = \langle \sin(yz), \cos(xz), 2 \rangle$
   (c) $\mathbf{F} = \langle \ln(xyz), \ln(yz), \ln z \rangle$

2. Determine whether $\mathbf{F}$ is conservative and compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.
   (a) $\mathbf{F} = \langle ye^{xy}, xe^{xy} \rangle$ and C is the path consisting of the line segment from $(0, 3)$ to $(0, 0)$ followed by the line segment from $(0, 0)$ to $(3, 0)$.
   (b) $\mathbf{F} = \langle y^2z^2 + 2x + 2y, 2xyz^2 + 2x, 2xy^2z + 1 \rangle$ and C consists of the two line segments $\overline{A_1A_2}$, $\overline{A_2A_3}$, where $A_1 = (1, 1, 1)$, $A_2 = (a, b, c)$ and $A_3 = (1, 2, 3)$.
   (c) $\mathbf{F} = \langle xz, yz, z^2 \rangle$ and C is the part of the helix $\mathbf{r}(t) = \langle 2\sin t, -2\cos t, t \rangle$ that lies inside the ellipsoid $x^2 + y^2 + 2z^2 = 6$.
   (d) $\mathbf{F} = \langle y - z^2, x + \sin z, y\cos z - 2xz \rangle$ and C is one turn of a helix of radius 2 from $(2, 0, 0)$ to $(2, 0, 1)$.
   (e) $\mathbf{F} = \left\langle 2(y+z)^{1/2}, -x(y+z)^{3/2}, -x(y+z)^{-3/2} \right\rangle$ and C is a smooth curve from the point $(1, 1, 3)$ to the point $(2, 4, 5)$.

3. Evaluate $\int_C y^2\, dx + 2xy\, dy$, where C is the semicircular arc $y = \sqrt{1 - x^2}$ from the $(-1, 0)$ to $(0, 1)$.

4. Evaluate $\int_C (2x - 3y + 1)\, dx - (3x + y - 5)\, dy$, where C is the curve $y = e^x$ from $(0, 1)$ to $(2, e^2)$.

5. Evaluate $\int_C (2x - 3y + 1)\, dx - (3x + y - 5)\, dy$, where C is the closed curve consisting of the segment from $(0, 1)$ to $(0, -1)$ followed by the semicircular arc $x = \sqrt{1 - y^2}$ from $(0, -1)$ to $(0, 1)$.

6. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle 2xy, x^2 + y^2 \rangle$ and C is the ellipse $(x/5)^2 + (y/4)^2 = 1$ from $(5, 0)$ to $(0, 4)$.

7. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle y\sin z, x\sin z, xy\cos x \rangle$ and C is parameterized by $\langle t^2, t^2, 0 \rangle$ for $0 \le t \le 2$.

4.3 Green's Theorem

Definition 4.3.1. let C be a simple closed curve that encloses a region D. We say C is positively oriented relative to D if D lies to the left of C as C is traversed.

Here are some pictures of curves and their induced orientations:

Theorem 4.3.2 (**Green's Theorem**). *Let D be a region in the plane whose boundary ∂D is a positively oriented piecewise-smooth simple closed curve. If the component functions F_1 and F_2 of the vector field $\mathbf{F} = \langle F_1, F_2 \rangle$ have continuous first-order partials in an open region containing D then*

$$\int\int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial D} F_1 \, dx + F_2 \, dy.$$

Remark 4.3.3. By definition 4.2.16,

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{k}}.$$

Therefore, the equality in Green's Theorem may also be written

$$\iint_D \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{k}} \, dA = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}.$$

Example 4.3.4. Let C be the positively oriented boundary of the disk $x^2+y^2 \leq R^2$. Compute $\oint_C -x^2y\ dx + xy^2\ dy$ in two ways; directly and by using Green's Theorem.

Solution

Example 4.3.5. Use Green's Theorem to compute $\int_C x^2\,dx + x\,dy$, where C is the semicircular arc $x^2 + y^2 = ax$ traversed from $(a, 0)$ to $(0, 0)$. Note that C is not a closed curve as Green's Theorem requires.

Solution

Corollary 4.3.6 (Area as a Line Integral). *Suppose D is a planar region that satisfies the hypotheses of Green's Theorem. Then the area A of D is given by each of the three line integrals*

$$A = \oint_{\partial D} x\ dy = -\oint_{\partial D} y\ dx = \frac{1}{2}\oint_{\partial D} x\ dy - y\ dx.$$

Example 4.3.7. Use a line integral to compute the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Solution

Example 4.3.8. Use Green's Theorem to compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle 2y + \cos(x^2), x^2 + y^3 \rangle$ and C consists of the two line segments $(0,0) \to (1,1)$ and $(1,1) \to (0,2)$.

Solution

Example 4.3.9. Let $\mathbf{F} = \langle y^3, x^3 + 3xy^2 \rangle$. Use Green's Theorem to compute the work done by $\mathbf{F}$ in moving a particle once around a circle of radius 3 and center $(0, 0)$ in the clockwise direction. Can you determine based on the result of your computation whether $\mathbf{F}$ is conservative?

Solution

Example 4.3.10. Let D be the region inside the ellipse $x^2/9+y^2/4=1$ and outside the circle $x^2+y^2=1$. Use Green's Theorem to compute $\int_{\partial D} 2xy\, dx + (x^2+2x)\, dy$, where ∂D is assumed to be positively oriented.

Solution

4.3.1 Exercises

1. Let D be the square with vertices $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$. Compute both $\oint_{\partial D} y^2\ dx + x^2\ dy$ and $\int\int_D 2(x-y)\ dA$, where ∂D is positively oriented. Why are the results related?

2. Let D be the planar region bounded by the graphs of $y = x$ and $y = x^2/4$ and let ∂D be positively oriented. Compute both $\oint_{\partial D} y^2\ dx + x^2\ dy$ and $\int\int_D 2(x-y)\ dA$. Why are the results related?

3. Let C be the closed curve consisting of the line segments $(0,1) \to (0,0)$, $(0,0) \to (1,0)$ and the parabola $y = 1 - x^2$ traversed from $(1,0)$ to $(0,1)$. Compute $\oint_C 2xy\ dx + x^2\ dy$.

4. Use Green's Theorem to evaluate the line integral. Assume throughout that ∂D is positively oriented.

   (a) $\oint_{\partial D}(y-x)\ dx + (2x-y)\ dy$, where D is the region between the graphs of $y = x$ and $y = x^2 - x$.

   (b) $\oint_{\partial D}(x^2 - y^2)\ dx + 2xy\ dy$, where D is the region bounded by the circle $x^2 + y^2 = a^2$.

   (c) $\oint_{\partial D} 2\arctan(y/x)\ dx + \ln(x^2+y^2)\ dy$, where D is the region enclosed by the parametric curve $\mathbf{r}(t) = \langle 4 + 2\cos t, 4 + \sin t\rangle$ for $0 \le t \le 2\pi$.

   (d) $\oint_{\partial D} \sin x \cos y\ dx + (xy + \cos x \sin y)\ dy$, where D is the region enclosed by the graphs of $y = x$ and $y = \sqrt{x}$.

   (e) $\oint_{\partial D}(e^{-x^2/2} - y)\ dx + (e^{-y^2/2} + x)\ dy$, where D is the region between the circle parameterized by $\mathbf{r}_1(t) = \langle 6\cos t, 6\sin t\rangle$ and the ellipse parameterized by $\mathbf{r}_2(t) = \langle 3\cos t, 2\sin t\rangle$ for $0 \le t \le 2\pi$.

   (f) $\oint_{\partial D} x\sin(x^2)\ dx + (xy^2 - x^8)\ dy$, where D is the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

   (g) $\oint_{\partial D}(\sqrt{x} + y^3)\ dx + (x^2 + \sqrt{y})\ dy$, where D is the region bounded by the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$.

5. Let C_1 be the segment from $(1,1)$ to $(2,6)$ and let C_2 be the parabolic curve $y = x(2x-1)$ traversed from $(1,1)$ to $(2,6)$. Use Green's Theorem to compute the difference between $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F} = \langle (x+y)^2, -(x-y)^2\rangle$.

6. Let C be the upper part of the circle $x^2 + y^2 = ax$ oriented from $(a,0)$ to $(0,0)$. Use Green's Theorem to compute $\int_C (e^x \sin y - 2x)\ dx + (e^x \cos y - 2)\ dy$ (note that C is not a closed curve).

7. Use Green's Theorem to compute the work done by the force $\mathbf{F} = \langle 3xy^2 + y^3, y^4 \rangle$ in moving a particle along the circle $x^2 + y^2 = a^2$ from $(0, -a)$ to $(0, a)$ counterclockwise.

8. Use a line integral to compute the area of the planar region D.

   (a) D is the triangle bounded by the lines $x = 0$, $3x - 2y = 0$ and $x + 2y = 8$.

   (b) D is bounded by $y = 2x + 1$ and $y = 4 - x^2$.

   (c) D is the region enclosed by $x^{2/3} + y^{2/3} = a^{2/3}$.

4.4 Flux

Consider a flow of fluid. Suppose the fluid has constant velocity $\mathbf{v}$ and constant mass density σ (measured in kg/m^3). If a flat surface S of surface of area ΔA is held stationary in the fluid perpendicular to the direction of fluid flow, what is the time-rate of mass transfer across the S?

Answer: In a time interval of length Δt, all the mass in a cylinder of length $h = \|\mathbf{v}\| \Delta t$ and base area ΔA gets transferred across S. The volume of this cylinder is $\Delta V = \|\mathbf{v}\| \Delta t \Delta A$, so the total mass in the cylinder is

$$\Delta m = \sigma \Delta V = \sigma \|\mathbf{v}\| \Delta t \Delta A.$$

Therefore, the time-rate of mass transfer of mass across S is

$$\Delta \Phi = \frac{\Delta m}{\Delta t} = \sigma \|\mathbf{v}\| \Delta A.$$

Suppose now that S is flat but no longer perpendicular to the direction of fluid flow. What is the time-rate of mass transfer across S?

Answer: The answer depends on the angle θ between the unit normal vector $\widehat{\mathbf{n}}$ of S, and the velocity vector $\mathbf{v}$. In a time interval of length Δt, the mass in a cylinder of length $h = \|\mathbf{v}\| \,\Delta t$ and base area $\Delta A \cos\theta$ is transferred across S. The volume of this cylinder is

$$\Delta V = \|\mathbf{v}\| \,\Delta t \Delta A \cos\theta = \Delta t \Delta A \,\|\mathbf{v}\| \,\|\widehat{\mathbf{n}}\| \cos\theta = \mathbf{v} \cdot \widehat{\mathbf{n}} \;\Delta A \;\Delta t.$$

Therefore, the mass in the cylinder is

$$\Delta m = \sigma \Delta V = \sigma \mathbf{v} \cdot \widehat{\mathbf{n}} \;\Delta A \;\Delta t.$$

Dividing by Δt, we obtain the (approximate) time-rate of mass transfer across S

$$\Delta\Phi = \frac{\Delta m}{\Delta t} = \sigma \mathbf{v} \cdot \widehat{\mathbf{n}} \;\Delta A. \tag{4.2}$$

Finally, consider the case where each of the following hold.

1. S is a smooth surface, not necessarily flat.
2. The velocity $\mathbf{v}$ is a continuous vector field, not necessarily constant.
3. The density σ is continuous, not necessarily constant.

In this case, what is the time-rate of mass transfer across S?

To answer this, cut S into N 'small' subsurfaces $S_1, S_2, \cdots, S_N$ whose surface areas are $\Delta S_1, \Delta S_2, \cdots, \Delta S_N$ respectively. For each $i = 1, 2, \cdots, N$, we approximate the rate of mass transfer $\Delta\Phi_i$ across S_i as follows. Choose any $\mathbf{r}_i^* \in S_i$. Since each of $\mathbf{v}$ and σ are continuous on S, and since S_i is 'small enough', we may approximate the values of $\mathbf{v}$ and σ on S_i by the constants $\mathbf{v}(\mathbf{r}_i^*)$ and $\sigma(\mathbf{r}_i^*)$ respectively. Moreover, since S is smooth and since S_i is 'small enough', we may approximate S_i to be flat. Therefore, as in equation (4.2), the rate of mass transport across S_i is approximately

$$\Delta\Phi_i \approx \sigma(\mathbf{r}_i^*)\mathbf{v}(\mathbf{r}_i^*) \cdot \widehat{\mathbf{n}}\, \Delta S_i.$$

The total rate of mass transport across S is approximated by

$$\begin{aligned} \Phi &\approx \Delta\Phi_1 + \Delta\Phi_2 + \cdots + \Delta\Phi_N \\ &= \sum_{i=1}^{N} \Delta\Phi_i \\ &= \sum_{i=1}^{N} \sigma(\mathbf{r}_i^*)\mathbf{v}(\mathbf{r}_i^*) \cdot \widehat{\mathbf{n}}\, \Delta S_i. \end{aligned}$$

Letting $N \to \infty$ in a way that forces $\Delta S_i \to 0$ for each i, we get the actual value of the time-rate of mass transport across S

$$\Phi = \lim_{N\to\infty} \sum_{i=1}^{N} \sigma(\mathbf{r}_i^*)\mathbf{v}(\mathbf{r}_i^*) \cdot \widehat{\mathbf{n}}\, \Delta S_i = \int\int_S \sigma(x,y,z)\mathbf{v} \cdot \widehat{\mathbf{n}}\, dS,$$

provided the limit exists.

Remark 4.4.1. Suppose S is a surface with smooth parameterization

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, \qquad (u,v) \in D.$$

Each of the derivatives $\mathbf{r}_u$ and $\mathbf{r}_v$ are tangent to S. The vector $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$ is normal to S (recall that we require $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ in the definition of smooth surface, so we also have $\mathbf{n} \neq \mathbf{0}$).

Proposition 4.4.2. *If S is the graph of a smooth function $g(x,y)$ for $(x,y) \in D$, then a normal vector field $\mathbf{n}$ on S is*

$$\mathbf{n} = \langle -g_x, -g_y, 1\rangle.$$

Proof. Since S is the graph of $g(x,y)$ for $(x,y) \in D$, a parameterization of S is

$$\mathbf{r}(x,y) =$$

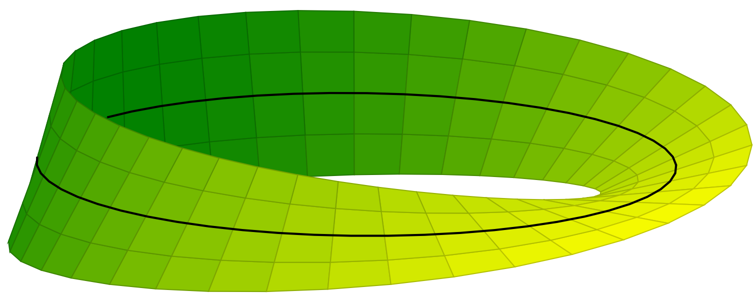
Therefore,

$$\mathbf{r}_x \times \mathbf{r}_y =$$

∎

Definition 4.4.3. A smooth surface S is called orientable if it is possible to determine a consistent choice of 'upward' (or 'downward' or 'outward' etc) normal vector. A surface S is called oriented if S is orientable and a choice of 'upward' normal vector has been chosen for S.

Example 4.4.4. The Möbius strip is not orientable.



Definition 4.4.5. Let S be a piecewise smooth oriented surface and let $\mathbf{F}$ be a vector field whose normal component $\mathbf{F} \cdot \widehat{\mathbf{n}}$ is integrable on S. The flux Φ of $\mathbf{F}$ across S is

$$\Phi = \int\int_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$$

Proposition 4.4.6. *Suppose S is a smooth orientable surface and $\mathbf{F}$ is a vector field on S such that the normal component $\mathbf{F} \cdot \widehat{\mathbf{n}}$ of $\mathbf{F}$ is integrable on S. If S has smooth parameterization $\mathbf{r}(u, v)$ for $(u, v) \in D$, then*

$$\begin{aligned}
\int\int_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS &= \int\int_D \mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\cancel{\|\mathbf{r}_u \times \mathbf{r}_v\|}} \cancel{\|\mathbf{r}_u \times \mathbf{r}_v\|}\, dA \\
&= \int\int_D \mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{r}_u \times \mathbf{r}_v\, dA.
\end{aligned}$$

Remark 4.4.7 (**Evaluating Flux Integrals on Smooth Surfaces**). Suppose S is a smooth orientable surface and $\mathbf{F}$ is a vector field on S such that the normal component $\mathbf{F}\cdot\widehat{\mathbf{n}}$ of $\mathbf{F}$ is integrable on S. To evaluate $\Phi = \int\int_S \mathbf{F}\cdot\widehat{\mathbf{n}}\, dS$, follow these steps.

1. Find a smooth parameterization

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle\,, \qquad (u,v)\in D$$

   for S. Be sure to include the parameter domain D.

2. Compute $\mathbf{r}_u \times \mathbf{r}_v$.

3. Substitute $x = x(u,v)$, $y = y(u,v)$ and $z = z(u,v)$ into $\mathbf{F}$'s formula according to your parameterization $\mathbf{r}(u,v)$.

4. Compute $\mathbf{F}(\mathbf{r}(u,v))\cdot\mathbf{r}_u\times\mathbf{r}_v$.

5. Evaluate the ordinary double integral $\int\int_D \mathbf{F}(\mathbf{r}(\mathbf{u},\mathbf{v}))\cdot\mathbf{r}_u\times\mathbf{r}_v\; dA$ (you may need to change variables in this step).

Remark 4.4.8 (**Evaluating Flux Integrals on Piecewise-Smooth Surfaces**). Suppose S is a piecewise-smooth oriented surface and that $\mathbf{F}$ is a vector field on S whose normal component $\mathbf{F}\cdot\widehat{\mathbf{n}}$ is integrable on S. To evaluate $\int\int_S \mathbf{F}\cdot\widehat{\mathbf{n}}\; dS$, complete the following steps.

1. Cut S into pieces $S_1, S_2, \cdots S_m$, each of which is a smooth surface.

2. For each subsurface S_i found in step 1, compute $\int\int_{S_i}\mathbf{F}\cdot\widehat{\mathbf{n}}\; dS$ according to the steps in remark 4.4.7.

3. Use additivity of the surface integral to recover the flux of $\mathbf{F}$ across the entire surface S as follows

$$\Phi = \int\int_S \mathbf{F}\cdot\widehat{\mathbf{n}}\; dS = \sum_{i=1}^{m}\int\int_{S_i}\mathbf{F}\cdot\widehat{\mathbf{n}}\; dS.$$

Example 4.4.9. Evaluate the upward flux of $\mathbf{F} = \langle xz, yz, z \rangle$ across the part of the paraboloid $z = 1 - x^2 - y^2$ that lies in quadrant one.

Solution

Example 4.4.10. Compute the outward flux of the vector field $\mathbf{F} = \langle xz^2, yz^2, z^3 \rangle$ across the sphere $x^2 + y^2 + z^2 = 1$.

Solution

Example 4.4.11. Compute the inward flux of $\mathbf{F} = \langle 3z, -4, y \rangle$ across the boundary of the solid that lies inside the cylinder $x^2 + y^2 = 1$ and between the planes $z = 0$ and $z = 1 - y$.

Solution

4.4.1 Exercises

1. Compute the upward flux of the given vector field $\mathbf{F}$ across the specified surface S.
   (a) $\mathbf{F} = \langle 3z, -2, y\rangle$, S is the part of the plane $x + 2y + z = 1$ that lies in octant one.
   (b) $\mathbf{F} = \langle x, y, z\rangle$, S is the graph of the function $z = 9 - x^2 - y^2$ corresponding to $z \geq 0$.
   (c) $\mathbf{F} = \langle x, y, -2z\rangle$, S is the graph of the function $z = \sqrt{a^2 - x^2 - y^2}$.
   (d) $\mathbf{F} = \langle xy, xz, xy\rangle$, S is the part of the graph of the function $z = 1 - x^2 - y^2$ corresponding to $0 \leq x \leq 1$, $0 \leq y \leq 1$.

2. Compute the flux of the given vector field $\mathbf{F}$ across the specified oriented surface S.
   (a) $\mathbf{F} = \langle 4xy, z^2, yz\rangle$, S is the outward-oriented cube in octant one with one vertex at $(0, 0, 0)$ and the opposite vertex at $(1, 1, 1)$.
   (b) $\mathbf{F} = \langle x + y, y, z\rangle$, S is the outward-oriented closed surface consisting of the graph of $z = 1 - x^2 - y^2$ and $z = 0$.
   (c) $\mathbf{F} = \langle y, -x, z^2\rangle$, S is the part of the downward-oriented paraboloid $z = 25 - x^2 - y^2$ corresponding to $z \geq 0$.
   (d) $\mathbf{F} = \langle xz, yz, z^2\rangle$, S is the part of the downward-oriented cone $z = \sqrt{x^2 + y^2}$ that lies below the plane $z = 1$ and in the first octant.
   (e) $\mathbf{F} = \langle xy, yz, x\rangle$, S is the part of the upward-oriented plane $2x-2y-z = 4$ that lies inside the cylinder $x^2 + y^2 = 1$.
   (f) $\mathbf{F} = \langle x, y, z\rangle / (x^2+y^2+z^2)^{3/2}$, S is the sphere of radius a oriented inward.
   (g) $\mathbf{F} = \langle y, x, z\rangle$, S is the boundary of the solid upper half ball $z \leq \sqrt{4 - x^2 - y^2}$.
   (h) $\mathbf{F} = \langle y - z, z - x, x - y\rangle$, S is the part of the cone $z = \sqrt{x^2 + y^2}$ below $z = 5$. S is oriented away from the z-axis.

4.5 Stokes' Theorem

Definition 4.5.1. Let C be an oriented simple closed curve and let $\mathbf{F}$ be a vector field on C. The circulation of $\mathbf{F}$ along C is the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Definition 4.5.2 (Induced orientation). Let S be an oriented surface with normal field $\widehat{\mathbf{n}}$ and suppose the boundary ∂S of S is a simple closed piecewise-smooth curve. Choose a normal vector on S whose direction agrees with the orientation of S and consider a small loop on S around the base point of the chosen normal vector. Orient this loop so that the direction of traversal of the loop agrees with both the direction of the chosen normal vector and the right-hand rule. Now continuously stretch this loop while staying on S until the loop is on ∂S. This process gives ∂S an orientation. We say ∂S is *positively oriented relative to* S if the direction of traversal of ∂S agrees with the normal field of S and the right-hand rule. In this case ∂S is said to have the *positive orientation induced by* S.

Theorem 4.5.3. *Let S be a piecewise-smooth oriented surface whose boundary ∂S is a simple closed curve. Suppose ∂S is positively oriented relative to S. If $\mathbf{F}$ is a vector field whose components have continuous partial derivatives in an open spatial region containing S, then*

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS. \tag{4.3}$$

Remark 4.5.4.

1. The integral on the left-hand side of equation (5.20) is the circulation of $\mathbf{F}$ along ∂S.
2. The integral on the right-hand side of equation (5.20) is the flux of $\mathbf{F}$'s curl through S.
3. Stokes Theorem says that under suitable hypotheses, if the orientation of S and ∂S agree, then the circulation of $\mathbf{F}$ along ∂S equals the flux of $\operatorname{curl} \mathbf{F}$ through S.

Remark 4.5.5 (**Using Stokes' Theorem to evaluate Type-II line integrals**). Suppose we want to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ along some simple, closed, piecewise-smooth curve C.

1. Choose *any* orientable surface S whose boundary is C. S should be chosen to be as simple as possible.
2. Choose an orientation of S that agrees with the orientation of C.
3. Compute $\operatorname{curl} \mathbf{F}$.
4. Compute the flux of $\operatorname{curl} \mathbf{F}$ via the surface integral $\iint_S \operatorname{curl} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$. If S is smooth, this can be achieved by following the steps in Remark 4.4.7 using $\operatorname{curl} \mathbf{F}$ in place of $\mathbf{F}$.

Example 4.5.6. Let C be the curve determined by the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. Suppose C is oriented clockwise as viewed from above. Compute the line integral of $\mathbf{F} = \langle xy, yz, xz \rangle$ along C.

Solution

Example 4.5.7. Let C be the ellipse determined by the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1$ oriented clockwise as viewed from above. Use Stokes' Theorem to compute the circulation of $\mathbf{F} = \langle y - z, z - x, x - y \rangle$ along C.

Solution

Remark 4.5.8 (**Stokes' Theorem to compute Type-II surface integrals**). Let S be an oriented surface and let $\mathbf{G}$ be a vector field on S. Suppose we want to compute $\int\int_S \mathbf{G} \cdot \widehat{\mathbf{n}}\, dS$. Follow these steps.

1. Find a vector field $\mathbf{F}$ on S such that $\operatorname{curl} \mathbf{F} = \mathbf{G}$ (in most of the problems I give, $\mathbf{G}$ is given as the curl of another vector field $\mathbf{F}$, so this step will be unnecessary).

2. Find ∂S, the boundary of S and be sure that the orientations of S and ∂S agree.

3. Compute $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$. This can be done, for example, by following the steps presented in Remark 4.2.7.

Example 4.5.9. Let S be the part of the sphere $x^2+y^2+z^2 = 25$ that lies below the plane $z = 4$, oriented outward. Use Stokes' Theorem to compute $\int\int_S \operatorname{curl} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$ for $\mathbf{F} = \langle -y, x, xyz \rangle$.

Solution

Example 4.5.10. Let S be the graph of the paraboloid $z = x^2 + y^2$ corresponding to $z \leq 4$ and oriented upward.

1. Use Stokes' Theorem to evaluate $\int\int_S \langle -3xz^2, 0, z^3 \rangle \cdot \widehat{\mathbf{n}}\, dS$.
2. With no further computation, compute the downward flux of $\langle -3xz^2, 0, z^2 \rangle$ through the graph of the function $z = 6 - \sqrt{x^2 + y^2}$ corresponding to $z \geq 4$.

Solution

4.5.1 Exercises

1. Verify that the conclusion of Stokes' Theorem holds by computing both the circulation of $\mathbf{F}$ along ∂S and the flux of $\operatorname{curl}\mathbf{F}$ through S.

   (a) $\mathbf{F} = \langle 2z, x, y^2\rangle$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ corresponding to $z \geq 0$.

   (b) $\mathbf{F} = \langle y, -x, z\rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 2$ that lies above the plane $z = 1$.

   (c) $\mathbf{F} = \langle x, y, xyz\rangle$ and S is the part of the plane $2x + y + z = 4$ in octant one.

   (d) $\mathbf{F} = \langle y, z, x\rangle$ and S is the part of the plane $x+y+z = 0$ inside the sphere $x^2 + y^2 + z^2 = 1$.

2. Let C be the boundary of the triangle whose vertices are $A_1 = (a, 0, 0)$, $A_2 = (0, a, 0)$ and $A_3 = (0, 0, a)$. Compute the work done by the force $\mathbf{F} = \langle y^2, z^2, x^2\rangle$ in moving a particle along the segments $A_1 \to A_2 \to A_3 \to A_1$.

3. Use Stokes' Theorem to compute the circulation of $\mathbf{F}$ along the closed curve C.

   (a) $\mathbf{F} = \langle x + y^2, y + z^2, z + x^2\rangle$ and C is the triangle traversed as follows: $(1, 0, 0) \to (0, 1, 0) \to (0, 0, 1) \to (1, 0, 0)$.

   (b) $\mathbf{F} = \langle yz, 2xz, e^{xy}\rangle$ and C is the curve determined by the intersection of the cylinder $x^2+y^2 = 1$ and the plane $z = 3$ oriented clockwise as viewed from above.

   (c) $\mathbf{F} = \langle xy, 3z, 3y\rangle$ and C is the curve determined by the intersection of the cylinder $y^2 + z^2 = 1$ and the plane $x + y = 1$.

   (d) $\mathbf{F} = \langle z, y^2, 2x\rangle$ and C is the curve determined by the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 6$. C is oriented counterclockwise as viewed from above.

   (e) $\mathbf{F} = \langle yz^2/2, -xz^2/2, 0\rangle$ and C is the boundary of the part of the cone $z = 2-\sqrt{x^2 + y^2}$ that lies in quadrant one. C is oriented counterclockwise as viewed from above.

   (f) $\mathbf{F} = \langle y - z, -x, x\rangle$ and C is the intersection of the cylinder $x^2+y^2 = 1$and the paraboloid $z = x^2+(y-1)^2$. C is oriented counterclockwise as viewed from above.

   (g) $\mathbf{F} = \left\langle z^2 - y^2, -2xy^2, e^{\sqrt{z}} \cos z\right\rangle$ and C is the curve with parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, 8 - \cos^2 t - \sin t\rangle$ for $0 \leq t \leq 2\pi$. *Hint: Use the parameterization of C to deduce an equation of a surface whose boundary is C.*

4. Let S be the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$ and suppose S is oriented upward. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$ for $\mathbf{F} = \langle yz, x^2 z, yx\rangle$

5. Let $\mathbf{F} = \langle -y, x, z\rangle$ and let S be the part of the elliptic paraboloid $z = 7 - x^2 - 4y^2$ that lies above the plan $z = 3$. Suppose S is oriented upward. Use Stokes' Theorem to compute $\iint_S \operatorname{curl} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$.

6. Let S be the part of the sphere $x^2 + y^2 + z^2 = 1$ corresponding to $z \geq 0$ with upward orientation. Use Stokes' Theorem to evaluate $\iint_S \langle x^3 e^y, -3x^2 e^y, 0\rangle \cdot \widehat{\mathbf{n}}\, dS$. *Hint: Find a vector field* $\mathbf{F}$ *whose curl is* $\langle x^3 e^y, -3x^2 e^y, 0\rangle$ *then use an appropriate line integral.*

4.6 Divergence Theorem

Definition 4.6.1. If $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is a vector field whose components are differentiable, then the divergence of $\mathbf{F}$ is

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Example 4.6.2. Compute the divergence of the vector field

$$\mathbf{F} = \left\langle x^2 + \cos(yz), y + \sin(x^2 z), xyz \right\rangle.$$

Solution

Let $D \subset \mathbb{R}^2$ be a bounded region whose boundary ∂D is a piecewise-smooth curve. Suppose ∂D has positive orientation and that $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$ is a parameterization for ∂D. An outward normal vector on ∂D is

$$\mathbf{n} = \langle y'(t), -x'(t) \rangle .$$

Let $\mathbf{F} = \langle F_1, F_2 \rangle$ be a vector field defined in an open region containing D and consider the vector field $\mathbf{G} = \langle -F_2, F_1 \rangle$. Applying Green's Theorem (Theorem 4.3.2) to $\mathbf{G}$, we obtain

$$\begin{aligned}
\int\int_D \operatorname{div} \mathbf{F} \, dA &= \int\int_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) \, dA \\
&= \int\int_D \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right) \, dA \\
&= \oint_{\partial D} \mathbf{G} \cdot d\mathbf{r} \\
&= \oint_{\partial D} -F_2 \, dx + F_1 \, dy \\
&= \oint_{\partial D} \mathbf{F} \cdot \widehat{\mathbf{n}} \, \|\mathbf{r}'(t)\| \, dt \\
&= \oint_{\partial D} \mathbf{F} \cdot \widehat{\mathbf{n}} \, ds.
\end{aligned}$$

This says that the outward flux of $\mathbf{F}$ through ∂D equals the integral of $\operatorname{div} \mathbf{F}$ over D. The generalization of this to three dimensions is the Divergence Theorem.

Theorem 4.6.3 (Divergence Theorem). *Suppose E is a closed region in space whose boundary ∂E is a piecewise-smooth surface with outward orientation. If $\mathbf{F}$ is a vector field whose components have continuous partial derivatives in an open region containing E, then*

$$\iiint_E \operatorname{div} \mathbf{F} \, dV = \int\int_{\partial E} \mathbf{F} \cdot \widehat{\mathbf{n}} \, dS.$$

Remark 4.6.4. The Divergence Theorem says that the outward flux of $\mathbf{F}$ across ∂E equals the triple integral of $\operatorname{div} \mathbf{F}$ over E.

Example 4.6.5. Let E be the part of the ball $x^2 + y^2 + z^2 \leq R^2$ in octant one. Compute the outward flux of

$$\mathbf{F} = \left\langle 4xy^2z + e^z,\ 4x^2yz,\ z^4 + \sin(xy)\right\rangle$$

across ∂E.

Solution

Corollary 4.6.6. *Suppose E is a closed solid region whose boundary ∂E is the union of two disjoint surfaces S_1 and S_2. Under the hypotheses of the Divergence Theorem,*

$$\iint_{S_1} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS = \iiint_E \operatorname{div} \mathbf{F}\, dV - \iint_{S_2} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS.$$

Example 4.6.7. Compute the upward flux of

$$\mathbf{F} = \left\langle z^2 \arctan(1 + y^2),\ z^4 \ln(1 + x^2),\ z \right\rangle$$

across the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$.

Solution

4.6.1 Exercises

1. Compute the divergence of the given vector field $\mathbf{F}$.
   (a) $\mathbf{F} = \langle x - 3y^2z, e^{xz} - y, 2z - \cos(xy)\rangle$.
   (b) $\mathbf{F} = \langle xy, y^2 + \sin(xz), \cos(xy)\rangle$.
   (c) $\mathbf{F} = \nabla f$, where $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$, for $(x,y,z) \neq (0,0,0)$.
   (d) $\mathbf{F} = \operatorname{curl}\mathbf{A}$, where $\mathbf{A} = \langle A_1, A_2, A_3\rangle$.

2. Use the Divergence Theorem to compute the outward flux of $\mathbf{F}$ through the specified surface S.
   (a) $\mathbf{F} = \langle xy^2, xz, x^2z\rangle$ and S is the boundary of the solid region inside the cylinder $x^2 + y^2 = 4$ and between the paraboloids $z = x^2 + y^2$ and $z = 1 + x^2 + y^2$.
   (b) $\mathbf{F} = \langle 2xy, 3ye^z, x\sin z\rangle$ and S is the cube of side-length one with one vertex at $(0,0,0)$ and the opposite vertex at $(1,1,1)$.
   (c) $\mathbf{F} = \langle xy, yz^2xz\rangle$ and S is the boundary of the solid region inside the cylinder $x^2+y^2 = 4$ and between the planes $z = -1$ and $z = 1$.
   (d) $\mathbf{F} = \langle yz, xz^2 + y, z - xz\rangle$ and $S = \partial E$, where E is the solid region bounded by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 1$.
   (e) $\mathbf{F} = \left\langle xz\sin(yz) + x^3, \cos(yz), 3zy^2 - e^{x^2+y^2}\right\rangle$ and $S = \partial E$, where E is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and plane $z = 0$.
   (f) $\mathbf{F} = \langle x - 3y^2z, e^{xz} - y, 2z\cos(xy)\rangle$ and $S = \partial E$, where E is the ball $x^2+y^2+z^2 \le R^2$.
   (g) $\mathbf{F} = \langle \tan(yz), \ln(1 + x^2z^2), z^2 + e^{xy}\rangle$ and S is the boundary of the smaller part of the ball $x^2+y^2+z^2 \le R^2$ between the two half-planes $y = x/\sqrt{3}$ and $y = \sqrt{3}x$ corresponding to $x \ge 0$.
   (h) $\mathbf{F} = \langle x + \tan(yz), \cos(xz) - y, \sin(xy) + z\rangle$ and $S = \partial E$, where E is the solid region bounded by the sphere $x^2+y^2+z^2 = 2z$ and the cone $z = \sqrt{x^2+y^2}$.

3. Use the technique of Example 4.6.7 to compute the flux of the given vector field across the given surface.
   (a) $\mathbf{F} = \langle xy^2, yz^2, y^2z + x^2\rangle$ and S is the top half of he sphere $x^2+y^2+z^2 = 4$ oriented toward the origin.
   (b) $\mathbf{F} = \langle z\cos(y^2), z^2\ln(1+x^2), z\rangle$ and S is the part of the paraboloid $z = 3 - x^2 - y^2$ that lies above the plane $z = 2$.

(c) $\mathbf{F} = \langle yz, xz, xy\rangle$ and S is the part of the cylinder $x^2 + y^2 = a^2$ corresponding to $0 \le z \le b$ oriented outward from the z-axis.

(d) $\mathbf{F} = \langle yz + x^3, x^2z^3, xy\rangle$ and S is the part of the cone $z = 1 - \sqrt{x^2 + y^2}$ corresponding to $z \ge 0$ oriented upward.

4.7 Final Exam Review Problems

Problem 4.1. Determine the mass m of one turn of the helix $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$, $0 \le t \le 2\pi$ if its mass density is $\sigma(\mathbf{r}) = \|\mathbf{r}\|^2$.

Problem 4.2. Evaluate the surface integral

$$\int\int_S (1 + x^2 + y^2)^{1/2}\, dS,$$

where S is the helicoid parameterized by

$$\mathbf{r}(u, v) = (u\cos v, u\sin v, v) \qquad \text{for} \quad (u, v) \in D = [0, 1] \times [0, 2\pi].$$

Problem 4.3. Let $E = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$ and consider the vector field $\mathbf{F} : E \to \mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = \left\langle -\frac{yz}{x^2}, \frac{z}{x}, \frac{y}{x} \right\rangle.$$

1. Show that $\mathbf{F}$ is conservative in E and find a scalar potential function U for $\mathbf{F}$ on E.

2. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is a piecewise smooth curve in E with initial point $(1, 2, 3)$ and final point $(3, 2, 1)$.

3. What is the value of the line integral of $\mathbf{F}$ along any closed piecewise smooth curve lying entirely in E?

Problem 4.4. Find the work W done by the force $\mathbf{F} = \langle y, xz, x\rangle$ in moving a point object along the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the clockwise direction.

Problem 4.5. Let C be the circle $x^2 + y^2 = R^2$ traversed once in the counterclockwise direction.

1. Evaluate the line integral

$$\int_C -x^2y\,dx + xy^2\,dy$$

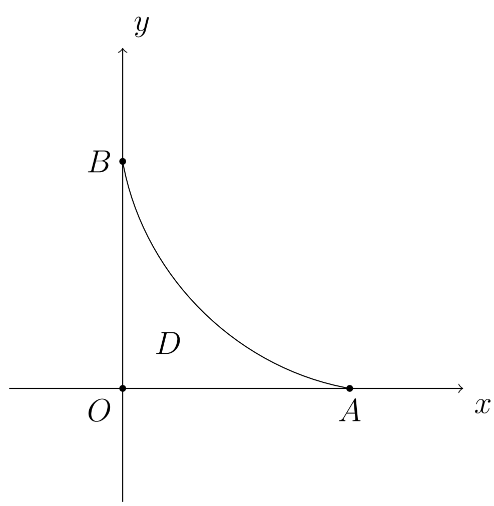
directly (i.e. do not use Green's Theorem).

2. Evaluate the line integral $\int_C -x^2y\,dx + xy^2\,dy$ using Green's Theorem.

Problem 4.6. Use a line integral to find the area $\mathcal{A}$ of the region D which is located in the first quadrant and is bounded by the curve

$$x^{1/2} + y^{1/2} = 1$$

and the segments OA and OB as shown in the figure.



Problem 4.7. Use Stokes' Theorem to evaluate the circulation of the vector field $\mathbf{F} = \langle y, -x, z\rangle$ along the curve C given by the intersection of the cylinder $(x-1)^2 + y^2 = 1$ and the hemisphere $x^2+y^2+z^2 = 4$, $z \geq 0$. Assume C is oriented counterclockwise as viewed from the top of the z-axis.

Problem 4.8. Consider the vector field $\mathbf{F} = \langle xy^2, x^2y, z\rangle$ and the surface S which is the part of the paraboloid $z = x^2+y^2$ that lies below the plane $z = 1$. Suppose S is oriented downward.

1. Compute the flux Φ of $\mathbf{F}$ across S via direct computation (i.e. use a surface integral involving $\mathbf{F}$).

2. Use the Divergence Theorem to compute the downward flux of $\mathbf{F}$ through S.

Problem 4.9. Let S be the boundary of the smaller domain enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$. Use the Divergence Theorem to evaluate the outward flux of $\mathbf{F} = \langle x^2, y^2, z \rangle$ across S. Graph S and indicate the unit normal vector which corresponds to the orientation of S.

Problem 4.10. If E is a solid region whose boundary is a closed surface S, the volume V of E may be represented by the surface integral

$$V = \frac{1}{3} \int \int_S \mathbf{r} \cdot \widehat{\mathbf{n}} \, dS.$$

Suppose S_1 is the surface parameterized by

$$\mathbf{r}(u, v) = \left\langle u \cos v, u \sin v, u^2 \right\rangle \qquad 0 \leq u \leq 1, 0 \leq v \leq 2\pi$$

with orientation $\widehat{\mathbf{n}} = -\mathbf{r}_u \times \mathbf{r}_v / \left\| \mathbf{r}_u \times \mathbf{r}_v \right\|$. Suppose S_2 is the upward-oriented surface given by

$$S_2 = \{(x, y, z) : x^2 + y^2 \leq 1, z = 1\}.$$

If E is the solid whose boundary is the closed surface $S = S_1 \cup S_2$, use the above surface integral to compute the volume V of E.

Chapter 5

Appendix: Solutions to Exam Review Problems

5.1 Exam 1 Review Solutions

Problem 5.1. The following are the equations of three distinct planes

$$2x - 3y + 6z = 1, \qquad 3x - 2y - 6z = 2, \qquad \text{and} \qquad -4x + 6y - 12z = 8.$$

[(a)]

1. Two of the planes are parallel. Determine which two and find the distance between them.
   Solution: Let $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ denote the planes corresponding to the first, second and third equation above respectively. The vectors

   $$\mathbf{n}_1 = \langle 2, -3, 6\rangle, \quad \mathbf{n}_2 = \langle 3, -2, -6\rangle, \quad \text{and } \mathbf{n}_3 = \langle -4, 6, -12\rangle$$

   are normal to $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ respectively. Since $\mathbf{n}_3 = -2\mathbf{n}_1$, $\mathcal{P}_1$ and $\mathcal{P}_3$ are parallel. To find the distance D between $\mathcal{P}_1$ and $\mathcal{P}_3$, first write the equations for $\mathcal{P}_1$ and $\mathcal{P}_3$ in the form

   $$\mathbf{n}_1 \cdot \langle x, y, z\rangle = d_1 \qquad \text{and} \qquad \mathbf{n}_1 \cdot \langle x, y, z\rangle = d_2 \tag{5.1}$$

   respectively, then use the equality

   $$D = \frac{|d_2 - d_1|}{\|\mathbf{n}_1\|}. \tag{5.2}$$

   Notice in particular, that the same normal vector is used for both equations in (5.1). The given equation for $\mathcal{P}_1$ is already in the desired form, so we have $d_1 = 1$. Dividing the given equation for $\mathcal{P}_3$ by -2 gives

   $$2x - 3y + 6z = -4,$$

   so $d_2 = -4$. Finally, since $\|\mathbf{n}_1\| = 7$ equation (5.2) yields

   $$D = \frac{|-4-1|}{7} = \frac{5}{7}.$$

2. Find the angle $\theta \in (0, \pi/2]$ between any two of the non-parallel planes.
   Solution: Planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are non-parallel. The non-obtuse angle between $\mathcal{P}_1$ and $\mathcal{P}_2$ is the same as the non-obtuse angle between the vectors

   $$\mathbf{n}_1 = \langle 2, -3, 6\rangle \qquad \text{and} \qquad \mathbf{n}_2 = \langle 3, -2, -6\rangle$$

which are normal to $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. The angle $\theta \in (0, \pi/2]$ between $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by

$$\begin{aligned}\cos\theta &= \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \\ &= \frac{|\langle 2, -3, 6\rangle \cdot \langle 3, -2, -6\rangle|}{\sqrt{49}\sqrt{49}} \\ &= \frac{24}{49}.\end{aligned}$$

Therefore,

$$\theta = \arccos\left(\frac{24}{49}\right).$$

Problem 5.2. The following equations represent three distinct lines.

$$\frac{x-1}{3} = \frac{y}{4} = \frac{z+5}{12}, \qquad x = 3t+1, y = 4t-2, z = 12t, \qquad \mathbf{r}(t) = \langle 2t+1, 3t-2, 6t\rangle.$$

Determine which of the lines are parallel and find the distance D between them.
Solution: Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ denote the lines represented by the first, second and third collection of equations above respectively. The vector equations for $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are

$$\mathbf{r}_1(t) = \langle 1, 0, -5\rangle + t\langle 3, 4, 12\rangle \qquad \mathbf{r}_2(t) = \langle 1, -2, 0\rangle + t\langle 3, 4, 12\rangle$$

and

$$\mathbf{r}_3(t) = \langle 1, -2, 0\rangle + t\langle 2, 3, 6\rangle$$

respectively, where $-\infty < t < \infty$. Since the direction vectors for $\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel (in fact, they're equal), $\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel. The distance between $\mathcal{L}_1$ and $\mathcal{L}_2$ is given by

$$D = \frac{\left\|\vec{AB} \times \vec{AC}\right\|}{\left\|\vec{AB}\right\|}, \tag{5.3}$$

where

$$A = (1, 0, -5) \qquad \text{and} \qquad B = (4, 4, 7)$$

are on $\mathcal{L}_1$ (corresponding to $t = 0$ and $t = 1$ respectively) and

$$C = (1, -2, 0)$$

is on $\mathcal{L}_2$ (corresponding to $t = 0$). Therefore,

$$\vec{AB} = \langle 3, 4, 12\rangle \qquad \text{and} \qquad \vec{AC} = \langle 0, -2, 5\rangle.$$

Using (5.3) and performing routine computations (the details of which have been omitted), we obtain

$$D = \frac{\sqrt{2197}}{13}.$$

Problem 5.3. The motion of an object is described by the function

$$\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle \qquad \text{for } \ t \geq 0.$$

1. Find both the velocity $\mathbf{v}(t)$ and the speed $\upsilon(t)$. Compute both $\lim_{t\to\infty} \mathbf{r}(t)$ and $\lim_{t\to\infty} \upsilon(t)$.
   Solution: The velocity is simply the componentwise derivative of the position and the speed is simply the magnitude of the velocity. Therefore,

   $$\mathbf{v}(t) = \langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), -e^{-t}\rangle$$

   and

   $$\begin{aligned} \upsilon(t) &= \left(e^{-2t}(\cos t + \sin t)^2 + e^{-2t}(\cos t - \sin t)^2 + e^{-2t}\right)^{1/2} \\ &= e^{-t}\left(2(\cos^2 t + \sin^2 t) + 1\right)^{1/2} \\ &= \sqrt{3}e^{-t}. \end{aligned}$$

   The requested limits are computed componentwise and are given by

   $$\lim_{t\to\infty} \mathbf{r}(t) = \langle \lim_{t\to\infty} e^{-t}\cos t, \lim_{t\to\infty} e^{-t}\sin t, \lim_{t\to\infty} e^{-t}\rangle = \langle 0, 0, 0\rangle$$

   and

   $$\lim_{t\to\infty} \upsilon(t) = \lim_{t\to\infty} \sqrt{3}e^{-t} = 0,$$

   where the limits in the first two components of $\lim_{t\to\infty} \mathbf{r}(t)$ are computed using the squeeze theorem.

2. Find the arc length L of the objects trajectory from time $t = 0$ to time $t = 1$. Express the length of the objects trajectory as a function of t for $t \geq 0$ (i.e. find $L(t)$). Compute $\lim_{t\to\infty} L(t)$ and interpret what the limit represents.
   Solution: The length of the object's trajectory as a function of t is given by

   $$\begin{aligned} L(t) &= \int_0^t \|\mathbf{r}'(u)\| \; du \\ &= \int_0^t \sqrt{3}e^{-u} \, du \\ &= \sqrt{3}(1 - e^{-t}), \qquad (5.4) \end{aligned}$$

where the computation from part (a) for the object's speed has been used in the second equality. Therefore, the length of the object's trajectory from time $t = 0$ to time $t = 1$ is given by

$$L(1) = \sqrt{3}(1 - e^{-1}).$$

Finally, using equation (5.4) we have

$$\lim_{t\to\infty} L(t) = \sqrt{3}(1 - 0) = \sqrt{3}.$$

This limit represents the total length of the objects trajectory if the object were allowed to travel for an infinite amount of time.

Problem 5.4. A curve C is defined by the parametric equations

$$x = t, \quad y = t^2, \quad z = t^4 \qquad -\infty < t < \infty.$$

1. Find an equation of the osculating plane at the point $(1, 1, 1) \in C$.
   Solution: Let $\mathcal{P}$ denote the osculating plane for C at the point $(1, 1, 1)$. To find an equation for $\mathcal{P}$, we need both a point on $\mathcal{P}$ and a vector $\mathbf{n}$ which is normal to $\mathcal{P}$. The point $(1, 1, 1)$ is on $\mathcal{P}$ and corresponds to $t = 1$ in the parameterization of C. A normal vector to $\mathcal{P}$ is given by $\mathbf{r}'(1)\times\mathbf{r}''(1)$. Routine computations show that

   $$\mathbf{r}'(1) = \langle 1, 2, 4\rangle \qquad \text{and} \qquad \mathbf{r}''(1) = \langle 0, 2, 12\rangle.$$

   Therefore, after performing more routine computations we arrive at

   $$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 16, -12, 2\rangle = 2\langle 8, -6, 1\rangle.$$

   The desired normal vector is

   $$\mathbf{n} = \langle 8, -6, 1\rangle.$$

   Using this normal vector and the point $(1, 1, 1)$ in the equation $\mathbf{n} \cdot \langle x, y, z\rangle = \mathbf{n} \cdot \langle 1, 1, 1\rangle$, we get the following equation for $\mathcal{P}$

   $$8x - 6y + z = 3.$$

2. Compute each of the following at $(1, 1, 1)$: the curvature, the unit tangent vector $\widehat{\mathbf{T}}$ and the binormal vector $\widehat{\mathbf{B}}$.
   Solution: Many of these computations were done in part (a), we just need to

plug the computations from part (a) into the correct formulas. The curvature at the point $(1,1,1)$ (which corresponds to $t=1$) is

$$\kappa(1) = \frac{\|\mathbf{r}'(1)\times\mathbf{r}''(1)\|}{\|\mathbf{r}'(1)\|^3} = \frac{2\,\|\langle 8,-6,2\rangle\|}{(1+4+16)^{3/2}} = \frac{2\sqrt{101}}{21^{3/2}}.$$

The unit tangent vector is

$$\widehat{\mathbf{T}}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\langle 1,2,4\rangle}{\sqrt{21}}.$$

The binormal vector is

$$\widehat{\mathbf{B}}(1) = \frac{\mathbf{r}'(1)\times\mathbf{r}''(1)}{\|\mathbf{r}'(1)\times\mathbf{r}''(1)\|} = \frac{2\langle 8,-6,1\rangle}{2\sqrt{101}} = \frac{\langle 8,-6,1\rangle}{\sqrt{101}}.$$

3. Calculate the torsion τ at the point $(1,1,1)$
Solution: The torsion at time $t=1$ is given by the equation

$$\tau(1) = \frac{(\mathbf{r}'(1)\times\mathbf{r}''(1))\cdot\mathbf{r}'''(1)}{\|\mathbf{r}'(1)\times\mathbf{r}''(1)\|^2}.$$

Since

$$\mathbf{r}'''(1) = \langle 0,0,24t\rangle\big|_{t=1} = \langle 0,0,24\rangle,$$

using the computations for $\mathbf{r}'(1)\times\mathbf{r}''(1)$ from part (a) in the equation for τ yields

$$\tau(1) = \frac{\langle 8,-6,1\rangle\cdot\langle 0,0,24\rangle}{(2\sqrt{101})^2} = \frac{2\cdot 24}{4\cdot 101} = \frac{12}{101}.$$

4. (BONUS) Is the curve C smooth? Is the curve planar? Justify your answers.
Solution: The curve C is smooth because $\mathbf{r}'(t)$ exists for all $t\in(-\infty,\infty)$ and since there is no $t\in(-\infty,\infty)$ for which $\|\mathbf{r}'(t)\|=0$. The curve is nonplanar since the torsion is nonzero.

5.2 Exam 2 Review Solutions

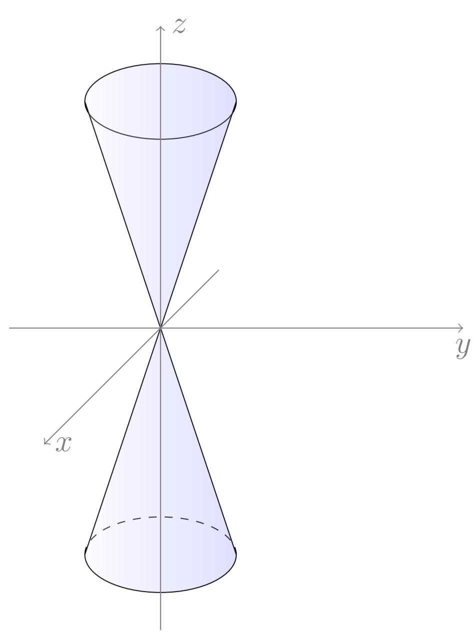
Problem 5.5. Find and graph the domain of the function

$$f(x,y,z) = \ln(z^2-x^2-y^2) + \sqrt{1-x^2-y^2-z^2}.$$

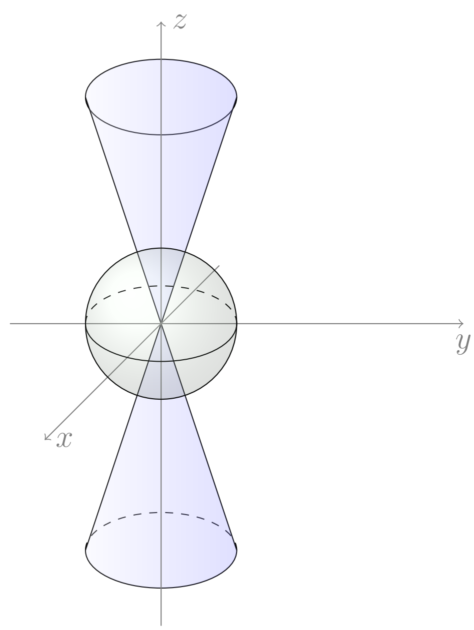
SolutionTo be in f's domain, (x, y, z) must simultaneously satisfy both

$$z^2 > x^2 + y^2 \qquad \text{and} \qquad x^2 + y^2 + z^2 \leq 1.$$

The first inequality represents the inside (without the boundary) of the double cone pictured below:



The second inequality represents the closed ball of radius 1 and center $(0, 0, 0)$. The graph of the f's domain is therefore the intersection of the inside of the ball with the inside of the cones:

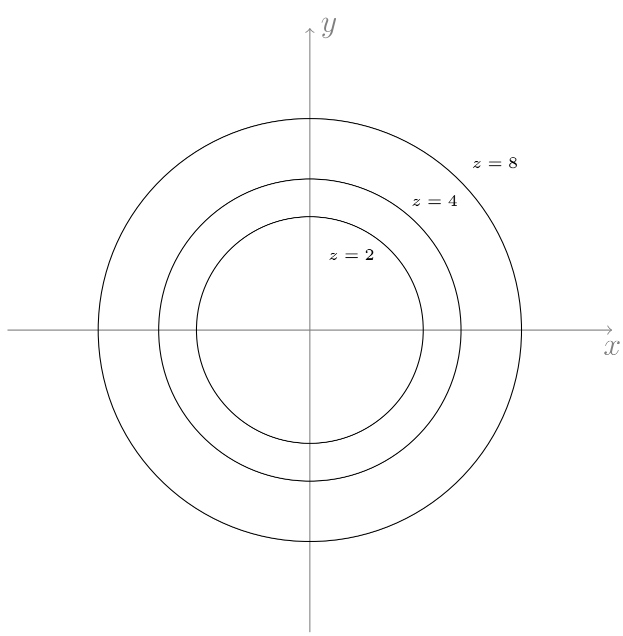


Problem 5.6. Consider the function $f(x, y) = 2x^2 + 2y^2$. Graph both f and the level curves of f corresponding to the levels $z = 2, 4, 8$.

SolutionThe level curves corresponding to $z = 2$, $z = 4$ and $z = 8$ are given by the equations

$$x^2 + y^2 = 1, \quad x^2 + y^2 = 2 \quad \text{and} \quad x^2 + y^2 = 4$$

respectively. The graph of the level curves is



Since f is an elliptic (in fact circular) paraboloid and the minimum of f occurs at $(0,0,0)$, I will let you draw the graph.

Problem 5.7. You must do this problem exactly the way I ask you to. Any other solutions will be given no credit. Let

$$f(x,y) = \sqrt{x^2+y^2}\sin(x^2+y^2).$$

1. Find a function g that depends only on $\|\mathbf{r}\| = \sqrt{x^2+y^2}$ such that both $|f(x,y)| \leq g(\|\mathbf{r}\|)$ and such that $\lim_{\mathbf{r}\to 0} g(\|\mathbf{r}\|) = 0$.
   SolutionSet

   $$f(x,y) = \sqrt{x^2+y^2}\sin(x^2+y^2) = \|\mathbf{r}\|\sin(\|\mathbf{r}\|^2).$$

   Since $\left|\sin(\|\mathbf{r}\|^2)\right| \leq 1$, we have

   $$\begin{aligned} 0 &\leq |f(\mathbf{r})| \\ &= \|\mathbf{r}\| \left|\sin(\|\mathbf{r}\|^2)\right| \\ &\leq \|\mathbf{r}\|, \end{aligned}$$

   so $g(\|\mathbf{r}\|) = \|\mathbf{r}\|$. Clearly, $\|\mathbf{r}\| \to 0$ as $\mathbf{r} \to 0$.

2. Use the function g you found in the previous part to compute $\lim_{\mathbf{r}\to\mathbf{0}} f(x,y)$.
   SolutionBy the inequalities obtained in part (a), and since $\lim_{\|\mathbf{r}\|\to 0} g(\mathbf{r}) = 0$, we have

   $$0 \leq |f(\mathbf{r}) - 0| \leq \|\mathbf{r}\| \to 0 \qquad \text{as } \|\mathbf{r}\| \to 0.$$

   Therefore, $\lim_{\|\mathbf{r}\|\to 0} f(\mathbf{r}) = 0$.

Problem 5.8. If it exists, find the value of a such that the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ a & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$. If no such a exists explain why.

SolutionNo such a exists. Indeed, if there were such an a, then it must be equal to $\lim_{(x,y)\to(0,0)} f(x,y)$. However, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. To see that the limit doesn't exist, consider the limits of f as $(x,y) \to (0,0)$ along the paths $y = mx$. We have

$$\begin{aligned} \lim_{(x,y)\to(0,0);y=mx} f(x,y) &= \lim_{x\to 0} f(x,mx) \\ &= \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} \\ &= \lim_{x\to 0} \frac{m}{1+m^2} \\ &= \frac{m}{1+m^2}, \end{aligned}$$

This computation implies that both

$$\lim_{(x,y)\to(0,0);y=x} f(x,y) = \frac{1}{2} \qquad \text{(by choosing } m=1\text{)}$$

and

$$\lim_{(x,y)\to(0,0);y=-x} f(x,y) = -\frac{1}{2} \qquad \text{(by choosing } m=-1\text{).}$$

Hence, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Problem 5.9. [(a)]

1. Prove that the function

$$f(x,y) = xy\sqrt{x^2+y^2}$$

is differentiable at $(0,0)$.

Solution #1: We must show that

$$(\star_1) \qquad \lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{\|(x,y)-(0,0)\|} = 0,$$

where $L(x,y)$ is the linearization of f at $(0,0)$. The linearization L of f at $(0,0)$ is

$$(\star_2) \qquad L(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y,$$

so we must compute each of $f(0,0)$, $f_x(0,0)$ and $f_y(0,0)$. The formula for f implies that $f(0,0) = 0$. To compute $f_x(0,0)$, we must use the limit definition of $f_x(0,0)$. This is because if you use usual rules of differentiation to compute a formula for f_x, the formula you get is not defined at $(0,0)$ [see solution #2 for the formula]. The limit definition of $f_x(0,0)$ is

$$(\star_3) \qquad \begin{aligned} f_x(0,0) &= \lim_{h\to 0} \tfrac{f(0+h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \tfrac{0-0}{h} \\ &= 0. \end{aligned}$$

Similarly, by using the limit definition of $f_y(0,0)$ we have $f_y(0,0) = 0$ (the details of this computation have been omitted). Returning to equation $(\star_2)$ we see that the linearization L of f at $(0,0)$ is $L(x,y) = 0$. Returning to $(\star_1)$ and using both $\|(x,y)\| = \sqrt{x^2+y^2}$ and $L(x,y) = 0$, we obtain

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\|(x,y) - (0,0)\|} &= \lim_{(x,y)\to(0,0)} \frac{xy\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} xy \\
&= 0,
\end{aligned}$$

so differentiability of f at $(0,0)$ is proven.

Solution #2: An alternative approach to showing that f is differentiable at $(0,0)$ is to show that f has first-order partials which are continuous at $(0,0)$. This approach works because if a function has first-order partials which are continuous at some point, then the function must be differentiable at that point. Below is an outline of the proof that f_x is continuous at $(0,0)$. The details of this proof as well as the proof that f_y is continuous at $(0,0)$ is left to you.

Performing routine computations (the details of which are omitted) we obtain

$$(\star_4) \qquad f_x(x,y) = \frac{y}{\sqrt{x^2+y^2}}\left(2x^2+y^2\right) \qquad \text{for } (x,y) \neq (0,0).$$

Using the limit definition of $f_x(0,0)$ as in $(\star_3)$ we have $f_x(0,0) = 0$. Finally, use $(\star_4)$ and a squeeze theorem argument to show that

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = 0 = f_x(0,0).$$

This computation shows that f_x is continuous at $(0,0)$.

2. Can we deduce based only on the result of part (a) that the first order partials of the given function are continuous?

   SolutionIf you proceeded as in solution #1 above, then the answer to this part is "no". In fact, an example of a function that is differentiable at $(0,0)$ but whose first-order partials are not continuous at $(0,0)$ is

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   If you proceeded as in solution #2 above then the answer to this part is "yes" because you would have proved continuity of the first-order partials in part (a).

Problem 5.10. Find the unit vector $\widehat{\mathbf{u}}$ in the direction of the steepest descent of the function $f(x, y) = x^3 + y^5$ at the point $(2, 1)$.

SolutionThe unit vector in the direction of greatest *ascent* is $\nabla f(2,1)/\,\|\nabla f(2,1)\|$. Therefore, $\widehat{\mathbf{u}} = -\nabla f(2,1)/\,\|\nabla f(2,1)\|$. Elementary computations show that

$$\nabla f(2,1) = \left.(3x^2, 5y^4)\right|_{(2,1)} = (12, 5)$$

and

$$\|\nabla f(2,1)\| = \sqrt{12^2 + 5^2} = 13.$$

Therefore,

$$\widehat{\mathbf{u}} = -\frac{1}{13}\,(12, 5).$$

Problem 5.11. [(a)]

1. Find an equation of the plane $\mathcal{P}$ tangent to the surface $x^2 + y^2 - z^2 = 4$ at the point $(2, 1, 1)$.
   SolutionSet $f(x, y, z) = x^2 + y^2 - z^2$. the gradient of f at $(2, 1, 1)$ is

   $$\nabla f(2,1,1) = \left.(2x, 2y, -2z)\right|_{(2,1,1)} = (4, 2, -2).$$

   Therefore, the equation of $\mathcal{P}$ is

   $$4(x-2) + 2(y-1) - 2(z-1) = 0,$$

   or

   $$2x + y - z = 4.$$

2. How does the gradient vector of the function $f(x, y, z) = x^2 + y^2 - z^2$ at the point $(2, 1, 1)$ relate to the level surface for f at that point?
   Solution$\nabla f(2, 1, 1)$ is orthogonal to the level surface $x^2 + y^2 - z^2 = 4$ at the point $(2, 1, 1)$.

Problem 5.12. Find all critical points of the function

$$f(x, y) = x^2 + y^2 - xy + 3x.$$

Use an appropriate test to determine whether each critical point is a local minimum, a local maximum or a saddle. Give the local minimum and local maximum values, if any, for f.

SolutionElementary computations yield

$$\begin{cases} f_x(x,y,z) & = 2x - y + 3 \\ f_y(x,y,z) & = 2y - x. \end{cases}$$

Solving the system

$$\begin{cases} f_x(x,y,z) & = 0 \\ f_y(x,y,z) & = 0 \end{cases}$$

yields one critical point, namely $(-2,-1)$. Moreover, $f(-2,-1) = -3$.
It remains to determine whether $(-2,-1)$ is a local maximizer, a local minimizer or a saddle. For this, we must compute $D(-2,-1)$, the determinant of the matrix of second-order partials of f at $(-2,-1)$. Routine computations yield

$$D(-2,-1) = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\Bigg|_{(-2,-1)} = \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3.$$

Since $D(-2,-1) > 0$ we must also consider the sign of $f_{xx}(-2,-1)$. In this case, $f_{xx}(-2,-1) = 2 > 0$, so by the second-derivative test, $(-2,-1)$ is a local minimizer.

Problem 5.13. Find all optimizers of

$$f(x,y) = xy + x + y$$

subject to the constraint $x^2y^2 = 4$.

SolutionLet $g(x,y) = x^2y^2 - 4$. The first step is to solve the system of equations

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = 0 \end{cases}$$

for x, y and λ. After performing routine computations, this system becomes

$$\begin{cases} y + 1 = 2\lambda x y^2 \\ x + 1 = 2\lambda x^2 y \\ x^2y^2 = 4. \end{cases} \tag{5.5}$$

Observe that no solution of (5.5) can have $x = 0$ (because the second equation would be violated) and that no solution of (5.5) can have $y = 0$ (because the first equation would be violated). Therefore, after dividing the first equation by xy^2 and dividing the second equation by x^2y, we obtain

$$\frac{y+1}{xy^2} = 2\lambda = \frac{x+1}{x^2y}.$$

After performing routine algebraic simplifications we get

$$xy(x-y) = 0.$$

Since neither x nor y are zero, we must have $x = y$. Returning to the third equation of (5.5) with this information we arrive at $x^4 = 4$. After solving this equation we get $x = \pm\sqrt{2}$. Combining this with the fact that $y = x$, the solutions of (5.5) are

$$(-\sqrt{2}, -\sqrt{2}) \qquad \text{and} \qquad (\sqrt{2}, \sqrt{2}).$$

Returning to f with these points, we get

$$f(-\sqrt{2}, -\sqrt{2}) = 2(1-\sqrt{2}) \qquad \text{and} \qquad f(\sqrt{2}, \sqrt{2}) = 2(1+\sqrt{2}).$$

We conclude that $(-\sqrt{2}, -\sqrt{2})$ is a minimizer of f subject to $g = 0$ and that $(\sqrt{2}, \sqrt{2})$ is a maximizer of f subject to $g = 0$.

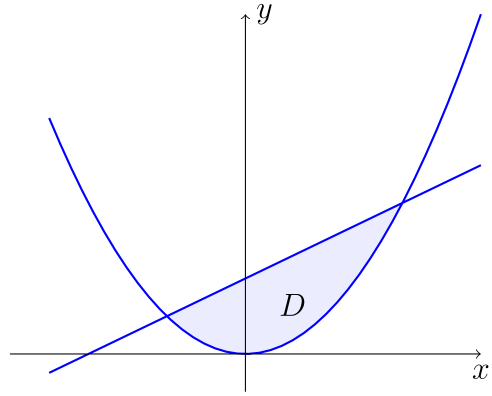
5.3 Exam 3 Review Solutions

Problem 5.14. Consider the iterated integral

$$\int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx.$$

[(a)]

1. Sketch the region whose area is represented by the iterated integral.
   Solution$D$ is the region enclosed by the line and the parabola shown below.



2. Evaluate the integral.
   Solution

$$\int_{-1}^{2}\int_{x^2}^{x+2} dy\ dx = \int_{-1}^{2}(x+2-x^2)\ dx = \frac{9}{2}.$$

3. Change the order of integration then evaluate the integral.
   SolutionThe given order of integration uses a description of D as a vertically simple region .To reverse the order of integration , we need to view D as a horizontally simple region. Unfortunately, D is not horizontally simple. To overcome this difficulty, split D into two regions D_1 and D_2 each of which is horizontally simple. Set

$$\begin{aligned} D_1 &= \{(x,y) : 0 \leq y \leq 1,\ -\sqrt{y} \leq x \leq \sqrt{y}\} \\ D_2 &= \{(x,y) : 1 \leq y \leq 4,\ y-2 \leq x \leq \sqrt{y}\}. \end{aligned}$$

The given integral is now evaluated as follows

$$\begin{aligned} \int_{-1}^{2}\int_{x^2}^{x+2} dy\ dx &= \int\int_{D} dA \\ &= \int_{D_1} dA + \int\int_{D_2} dA \\ &= \int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} dx\ dy + \int_{1}^{4}\int_{y-2}^{\sqrt{y}} dx\ dy \\ &= \int_{0}^{1} 2\sqrt{y}\ dy + \int_{1}^{4}(\sqrt{y}-y+2)\ dy \\ &= \frac{9}{2}. \end{aligned}$$

Problem 5.15. Let D be the region bounded by the curves

$$x = y^2, \quad 4x = y^2, \quad xy = 1 \quad \text{and} \quad xy = 2.$$

Use a double integral with a suitable change of variables to compute the area of D.

SolutionUse the change of variables

$$u = \frac{y^2}{x} \qquad v = xy, \tag{5.6}$$

so $T^{-1}(x, y) = (y^2/x, xy) = (u, v)$ is the transformation that expresses the new variables u and v as functions of the old variables x and y.The equations of the given curves imply that $1 \le u \le 4$ and $1 \le v \le 2$, so the uv-description of D is

$$D' = \{(u, v) : 1 \le u \le 4,\ 1 \le v \le 2\}.$$

The Jacobian of T^{-1} is

$$J_{T^{-1}}(x, y) = \det \begin{pmatrix} -\frac{y^2}{x^2} & \frac{2y}{x} \\ y & x \end{pmatrix} = -\frac{3y^2}{x} = -3u.$$

By the Inverse Function Theorem, the Jacobian of the transformation T that expresses x and y as functions of u and v is $J_T(u, v) = -\frac{1}{3u}$. Finally, the area of D is

$$\begin{aligned} A &= \int\int_D dA \\ &= \int\int_{D'} |J_T(u, v)|\ dA' \\ &= \int_1^2 \int_1^4 \frac{1}{3u}\, du\, dv \\ &= \frac{2}{3} \ln 2. \end{aligned}$$

Problem 5.16. Let D be the region bounded by the curves

$$x^2 + y^2 = 2x, \quad x^2 + y^2 = 4x, \quad y = -x, \quad \text{and} \quad y = x. \tag{5.7}$$

Suppose the mass-density of D at (x, y) is inversely proportional to the distance from (x, y) to the y-axis. Compute the center of mass of D.

SolutionFirst, we'll compute the mass $m = \int\int_D \sigma(x, y)\ dA$, where $\sigma(x, y) = k/x$ is the mass density and k is a positive constant. The will be accomplished by changing variables to polar coordinates. The first and second equations of (5.7) represent a circle of radius 1 with center $(1, 0)$ and a circle of radius 2 with center $(2, 0)$ respectively. Moreover, after converting to polar coordinates, the equations in (5.7) become

$$r = 2\cos\theta, \qquad r = 4\cos\theta, \qquad \theta = -\frac{\pi}{4} \qquad \text{and} \qquad \theta = \frac{\pi}{4}$$

respectively. Therefore, the polar-coordinate description of D is

$$D' = \{(r, \theta) : -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4},\ 2\cos\theta \leq r \leq 4\cos\theta\}.$$

The mass m is therefore,

$$\begin{aligned}
m &= \int\int_D \sigma(x, y)\ dA \\
&= k\int\int_D \frac{1}{x}\ dA \\
&= k\int\int_{D'} \frac{1}{r\cos\theta}\ r\ dA' \\
&= k\int_{-\pi/4}^{\pi/4}\int_{2\cos\theta}^{4\cos\theta} \frac{1}{\cos\theta}\ dr\ d\theta \\
&= k\pi.
\end{aligned}$$

The x-coordinate of the center of mass is

$$\begin{aligned}
\overline{x} &= \frac{1}{m}\int\int_D x\sigma(x, y)\ dA \\
&= \frac{k}{m}\int\int_{D'} r\ dA' \\
&= \frac{k}{m}\int_{-\pi/4}^{\pi/4}\int_{2\cos\theta}^{4\cos\theta} r\ dr\ d\theta \\
&= \frac{6k}{m}\int_{-\pi/4}^{\pi/4} \cos^2\theta\ d\theta \\
&= \frac{3}{2}\frac{\pi + 2}{\pi},
\end{aligned}$$

where $m = k\pi$ was used in the final equality.
The y-component of the center of mass is

$$\overline{y} = \frac{1}{m} \int\int_D y\sigma(x,y)\, dA.$$

The integral in this expression is computed as follows

$$\begin{aligned}
\int\int_D y\sigma(x,y)\, dA &= k \int\int_D \frac{y}{x}\, dA \\
&= k \int\int_{D'} r\tan\theta\, dA' \\
&= k \int_{-\pi/4}^{\pi/4} \int_{2\cos\theta}^{4\cos\theta} r\tan\theta\, dr\, d\theta \\
&= 6k \int_{-\pi/4}^{\pi/4} \cos^2\theta\tan\theta\, d\theta \\
&= 6k \int_{-\pi/4}^{\pi/4} \cos\theta\sin\theta\, d\theta \\
&= 0.
\end{aligned}$$

Therefore, $\overline{y} = 0$.

Problem 5.17. Consider the iterated integral

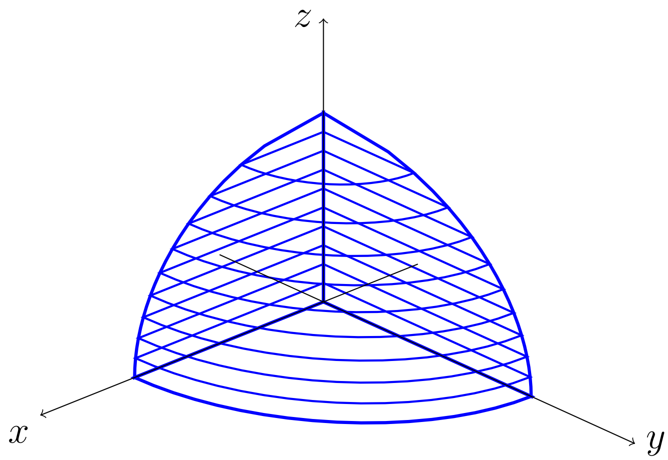
$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx,$$

where a is a positive constant.

1. Sketch the solid whose volume is represented by the iterated integral.
   SolutionLet E be the solid under consideration. The given integral is an iterated version of

   $$\int\int_D (f_{\text{upper}}(x,y) - f_{\text{lower}}(x,y))\, dA,$$

   where $f_{\text{upper}}(x,y) = \sqrt{a^2 - x^2 - y^2}$,$f_{\text{lower}}(x,y) = 0$, and $D = \{(x,y) : 0 \le x \le a,\ 0 \le y \le \sqrt{a^2 - x^2}\}$. Therefore, E is the part of the solid bounded by the sphere $x^2 + y^2 + z^2 = a^2$ that lies in octant one. Here is a sketch of E.



2. Evaluate the iterated integral. Check your answer by comparing with a suitable geometric formula.

SolutionTo evaluate the integral, change from rectangular coordinates to spherical coordinates.

$$\begin{aligned} \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx &= (\text{volume of } E) \\ &= \int\int\int_E dV \\ &= \iiint_{E'} \rho^2 \sin\phi\, dV', \qquad (5.8) \end{aligned}$$

where $E' = \{(\rho, \theta, \phi) : 0 \le \rho \le a,\ 0 \le \theta \le \pi/2,\ 0 \le \phi \le \pi/2\}$ is the region whose image under the polar-to-rectangular coordinate transformation is E. Continuing in equation (5.8) we have

$$\begin{aligned} \iiint_{E'} \rho^2 \sin\phi\, dV' &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\ &= \frac{1}{8}\left(\frac{4}{3}\pi a^3\right), \end{aligned}$$

which agrees with the usual formula for the volume of a sphere of radius a.

Problem 5.18. Let E be the volume of the solid that lies in octant one and is bounded by the spheres $x^2+y^2+z^2 = a^2$, $x^2+y^2+z^2 = b^2$, the cones $z = \sqrt{x^2+y^2}$, $z = \sqrt{3(x^2+y^2)}$ and the planes $x = 0$, $y = x$. Use a triple integral to compute the volume of E.

SolutionThe V denote the volume of E. Then

$$V = \iiint_E dV = \iiint_{E'} \rho^2 \sin\phi \, dV',$$

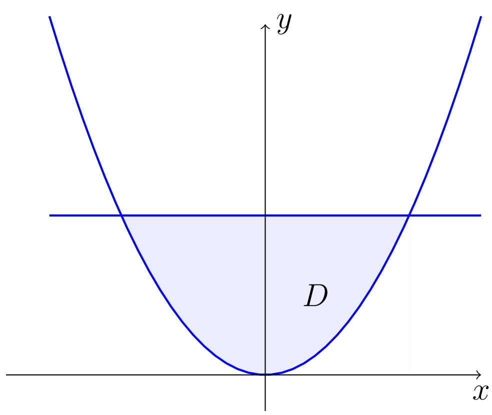
where $E' = \{(\rho, \theta, \phi) : a \le \rho \le b,\ \pi/4 \le \theta \le \pi/2,\ \pi/6 \le \phi \le \pi/4\}$. Therefore,

$$\iiint_{E'} \rho^2 \sin\phi \, dV' = \int_{\pi/6}^{\pi/4} \int_{\pi/4}^{\pi/2} \int_a^b \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \frac{\pi}{24}(b^3 - a^3)(\sqrt{3} - \sqrt{2}).$$

Problem 5.19. Let E be the solid bounded by

$$z = x^2 + y^2, \qquad y = x^2, \qquad y = 1 \qquad \text{and} \qquad z = 0.$$

1. Sketch both E and the projection of E onto the xy-plane.
   SolutionThe projection D of E onto the xy-plane is the region bounded by $y = x^2$ and $y = 1$ as shown below



   E is bounded above by the paraboloid $z = x^2 + y^2$ and below by $z = 0$ and is only above D. I will allow you to complete the sketch of E.

2. Suppose $f(x, y, z)$ is a function that is continuous on E. Express $\iiint_E f(x, y, z)\, dV$ as an iterated integral (you will not be able to evaluate because I didn't give you f).
   SolutionA description for E is

$$E = \{(x, y, z) : -1 \le x \le 1,\ x^2 \le y \le 1,\ 0 \le z \le x^2 + y^2\}.$$

Therefore,

$$\iiint_E f(x,y,z)\,dV = \int_{-1}^{1}\int_{x^2}^{1}\int_0^{x^2+y^2} f(x,y,z)\,dz\,dy\,dx.$$

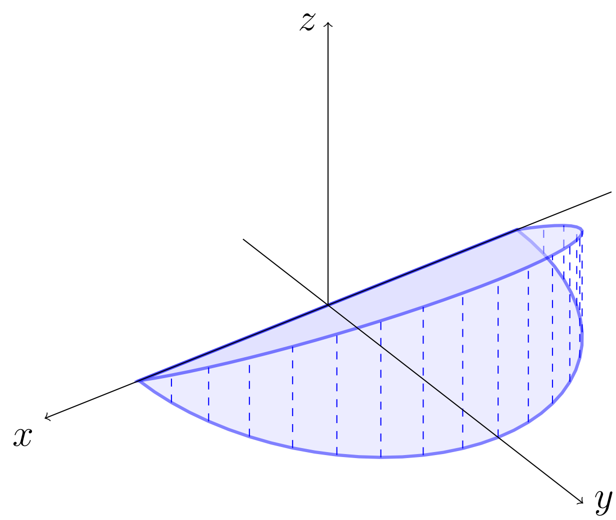
Problem 5.20. Let E be the solid bounded by

$$x^2+y^2=a^2, \qquad y=z, \qquad \text{and} \qquad z=0$$

corresponding to $z \geq 0$.

1. Sketch E and find the set E' whose image under the rectangular-to-cylindrical coordinate transformation is E.
   Solution$E$ is the wedge pictured below. A description of E in cylindrical coordinates is

$$E' = \{(r,\theta,z) : 0 \leq r \leq a,\ 0 \leq \theta \leq \pi,\ 0 \leq z \leq r\sin\theta\}.$$



2. Evaluate $\iiint_E y\,dV$.
   Solution

$$\begin{aligned}\iiint_E y\,dV &= \iiint_{E'} r^2\sin\theta\,dV' \\ &= \int_0^{\pi}\int_0^{a}\int_0^{r\sin\theta} r^2\sin\theta\,dz\,dr\,d\theta \\ &= \frac{\pi a^4}{8},\end{aligned}$$

where we have used $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$ for the evaluation of the integral.

Problem 5.21. Let D be the planar region which lies inside the curve

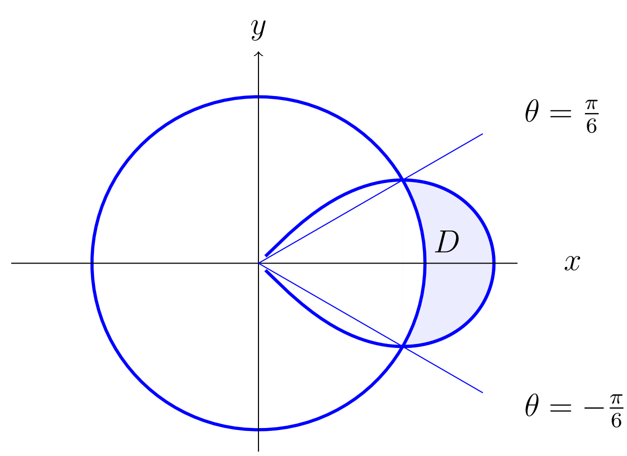
$$(x^2 + y^2)^2 = 4(x^2 - y^2),$$

and outside the curve $x^2 + y^2 = 2$ corresponding to $x \geq 0$.

1. Sketch D and find the region D' whose image under the rectangular-to-polar coordinate transformation is D.
   SolutionReplace x and y by $r \cos\theta$ and $r \sin\theta$ respectively in the given equations. For the first equation, using $\cos^2\theta - \sin^2\theta = \cos(2\theta)$ and $r \geq 0$ gives

   $$r = 2\sqrt{\cos(2\theta)}.$$

   The second equation becomes $r = \sqrt{2}$. To find the points of intersection of the curves whose equations are given, solve $\sqrt{2} = 2\sqrt{\cos(2\theta)}$ to get $\theta = \pm\pi/6$. Here is a sketch of D.

   The region D' whose image under the rectangular-to-polar coordinate transformation is D is

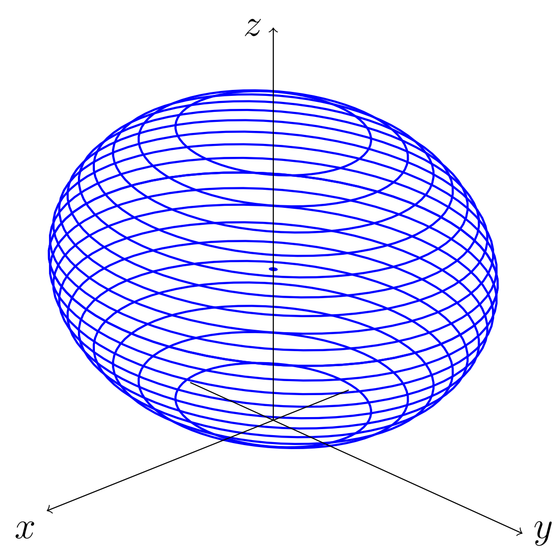
   $$D' = \left\{(r, \theta) : -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6},\ \sqrt{2} \leq r \leq 2\sqrt{\cos(2\theta)}\right\}.$$

2. Use a double integral to compute the area of D.
   SolutionThe area A of D is

$$\begin{aligned} A &= \int\int_D dA \\ &= \int\int_{D'} r\, dA' \\ &= \int_{-\pi/6}^{\pi/6}\int_{\sqrt{2}}^{2\sqrt{\cos(2\theta)}} r\, dr\, d\theta \\ &= \frac{1}{3}(3\sqrt{3}-\pi). \end{aligned}$$

Problem 5.22. Let $E = \{(x,y,z) : x^2+y^2+z^2 \leq 2z\}$.

1. Sketch E.
   SolutionThe defining inequality for E is equivalent to $x^2+y^2+(z-1)^2 \leq 1$, so E is a ball of radius 1 and center $(0,0,1)$.



2. If E has uniform mass-density, compute the moment of inertia of E about the z-axis.
   SolutionA description E' of E in cylindrical coordinates is

$$E' = \{(r,\theta,z) : 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ 1-\sqrt{1-r^2} \leq z \leq 1+\sqrt{1-r^2}\},$$

so since the mass density of E is a constant, say k,

$$
\begin{aligned}
I_z &= \iiint_E K(x^2+y^2)\,dV \\
&= k\int_0^{2\pi}\int_0^1\int_{1-\sqrt{1-r^2}}^{1+\sqrt{1-r^2}} r^2 r\,dz\,dr\,d\theta \\
&= 2k\int_0^{2\pi}\int_0^1 r^3\sqrt{1-r^2}\,dr\,d\theta \\
&= 2k\int_0^{2\pi}\int_0^1 r^3\sqrt{1-r^2}\,dr\,d\theta \\
&= 2k\int_0^{2\pi}\int_0^1 \frac{1}{2}(1-u)\sqrt{u}\,du\,d\theta \\
&= \frac{8\pi}{15},
\end{aligned}
$$

where we have used the substitution $u = 1 - r^2$ in the second-to last equality.

5.4 Final Exam Review Solutions

Problem 5.23. Determine the mass m of one turn of the helix $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$, $0 \le t \le 2\pi$ if its mass density is $\sigma(\mathbf{r}) = \|\mathbf{r}\|^2$.

SolutionThe quantity to be computed is $\int_C \sigma\,ds$, where C is the helix and

$$ds = \|\mathbf{r}'(t)\|\ dt = \|(-a\sin t, a\cos t, b)\|\ dt = \sqrt{a^2+b^2}\,dt.$$

is the length element on C. Using the given parameterization of C, we have

$$\sigma(\mathbf{r}) = \|\mathbf{r}(t)\|^2 = a^2 + b^2t^2.$$

Therefore,

$$
\begin{aligned}
\int_C \sigma\,dS &= \int_0^{2\pi}(a^2+b^2t^2)\sqrt{a^2+b^2}\,dt \\
&= \sqrt{a^2+b^2}\left(2\pi a^2 + \frac{b^2}{3}(2\pi)^3\right) \\
&= \frac{2\pi}{3}\sqrt{a^2+b^2}(3a^2+4b^2\pi^2).
\end{aligned}
$$

Problem 5.24. Evaluate the surface integral

$$\int\int_S (1 + x^2 + y^2)^{1/2}\, dS,$$

where S is the helicoid parameterized by

$$\mathbf{r}(u, v) = (u\cos v, u\sin v, v) \qquad \text{for} \qquad (u, v) \in D = [0, 1] \times [0, 2\pi].$$

SolutionLet $f(x, y) = (1 + x^2 + y^2)^{1/2}$. Then

$$\int\int_S f(x, y)\, dS = \int\int_D f(x(u, v), y(u, v))\ \|\mathbf{r}_u \times \mathbf{r}_v\|\ dA, \tag{5.9}$$

where $dA = du\, dv$ is the area element for D. Performing elementary computations gives

$$\mathbf{r}_u = (\cos v, \sin v, 0), \qquad \mathbf{r}_v = (-u\sin v, u\cos v, 1)$$

and

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|(\sin v, -\cos v, u)\| = \sqrt{1 + u^2}. \tag{5.10}$$

Moreover,

$$f(x(u, v), y(u, v)) = (1 + u^2\cos^2 v + u^2\sin^2 v)^{1/2} = (1 + u^2)^{1/2}. \tag{5.11}$$

Therefore, using equations (5.10) and (5.11) in equation (5.9), we get

$$\int\int_S f(x, y)\, dS = \int_0^{2\pi}\int_0^1 (1 + u^2)\, du\, dv = \frac{8\pi}{3}.$$

Problem 5.25. Let $E = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$ and consider the vector field $\mathbf{F} : E \to \mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = \left\langle -\frac{yz}{x^2}, \frac{z}{x}, \frac{y}{x} \right\rangle.$$

1. Show that $\mathbf{F}$ is conservative in E and find a scalar potential function U for $\mathbf{F}$ on E.
   SolutionWe seek a function $U : E \to \mathbb{R}$ such that $\nabla U = \mathbf{F}$. That is, U must simultaneously satisfy the following three conditions:

   $$\frac{\partial U}{\partial x} = -\frac{yz}{x^2}, \qquad \frac{\partial U}{\partial y} = \frac{z}{x}, \qquad \frac{\partial U}{\partial z} = \frac{y}{x}. \tag{5.12}$$

   Integrating the first equation with respect to x implies that

   $$U(x, y, z) = \frac{yz}{x} + f(y, z) \tag{5.13}$$

for some function f (which is independent of x but possibly dependent on both y and z). Similarly, the second and third equations in (5.12) imply

$$U(x,y,z) = \frac{yz}{x} + g(x,z) \tag{5.14}$$

and

$$U(x,y,z) = \frac{yz}{x} + h(x,y) \tag{5.15}$$

respectively, for some functions g and h. We want to choose f, g and h so that (5.13), (5.14) and (5.15) are simultaneously satisfied. It is easy to see that choosing $f = 0$, $g = 0$ and $h = 0$ is sufficient, so

$$U(x,y,z) = \frac{yz}{x}$$

is a potential for $\mathbf{F}$ on E. Since we have found a potential for $\mathbf{F}$ on E, $\mathbf{F}$ is conservative on E.

2. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is a piecewise smooth curve in E with initial point $(1,2,3)$ and final point $(3,2,1)$.
   SolutionSince $\mathbf{F}$ is conservative and has potential U (found in part (a)) in E, and since C is contained in E, the fundamental theorem of line integrals gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = U(3,2,1) - U(1,2,3) = \frac{2}{3} - 6 = -\frac{16}{3},$$

   where we have used $U(x,y,z) = (yz)/x$.

3. What is the value of the line integral of $\mathbf{F}$ along any closed piecewise smooth curve lying entirely in E?
   SolutionIf $C \subset E$ is any closed piecewise smooth curve then the final and initial points of C coincide. Therefore, by the fundamental theorem of line integrals,

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= U(\text{ final point of } C) - U(\text{ initial point of } C) \\ &= 0. \end{aligned}$$

Problem 5.26. Find the work W done by the force $\mathbf{F} = \langle y, xz, x \rangle$ in moving a point object along the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the clockwise direction.

SolutionThe work is computed via the formula

$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the given ellipse traversed once in the clockwise direction. The function

$$\mathbf{r}(t) = (a\cos t, b\sin t, 0) \qquad 0 \le t \le 2\pi$$

is a parameterization for $-C$ (because this parameterization traverses C in the counterclockwise direction). Therefore

$$W = -\int_{-C} \mathbf{F} \cdot d\mathbf{r}. \tag{5.16}$$

Now,

$$\mathbf{F}(\mathbf{r}(t)) = (b\sin t, 0, a\cos t) \quad \text{and} \quad \mathbf{r}'(t) = (-a\sin t, b\cos t, 0),$$

so

$$\mathbf{F} \cdot d\mathbf{r} = -ab\sin^2 t\ dt.$$

Returning to (5.16), we have

$$W = -\int_0^{2\pi} -ab\sin^2 t\ dt = \frac{ab}{2}\int_0^{2\pi} (1 - \cos(2t))\ dt = \pi ab.$$

Problem 5.27. Let C be the circle $x^2 + y^2 = R^2$ traversed once in the counterclockwise direction.

1. Evaluate the line integral

$$\int_C -x^2y\ dx + xy^2\ dy$$

directly (i.e. do not use Green's Theorem).
SolutionThe standard parameterization for C is

$$x(t) = R\cos t \qquad y(t) = R\sin t \qquad 0 \le t \le 2\pi.$$

For this parameterization we have

$$\begin{aligned} dx &= x'(t)\ dt = -R\sin t\ dt \\ dy &= y'(t)\ dt = R\cos t\ dt. \end{aligned}$$

Therefore, using the above parameterization, the trigonometric identities $2\sin\alpha\cos\alpha = \sin(2\alpha)$ and $\sin^2\alpha = (1-\cos(2\alpha))/2$, then performing elementary computations gives

$$\begin{aligned}
\int_C -x^2y\,dx + xy^2\,dy &= 2R^4\int_0^{2\pi}\cos^2 t\sin^2 t\,dt \\
&= 2R^4\int_0^{2\pi}\frac{1}{4}\left(2\cos t\sin t\right)^2\,dt \\
&= \frac{R^4}{2}\int_0^{2\pi}\sin^2(2t)\,dt \\
&= \frac{R^4}{4}\int_0^{2\pi}(1-\cos(4t))\,dt \\
&= \frac{1}{2}\pi R^4.
\end{aligned}$$

2. Evaluate the line integral $\int_C -x^2y\,dx + xy^2\,dy$ using Green's Theorem.
   SolutionLet $\mathbf{F}(x,y) = (-x^2y, xy^2)$ and let D denote the interior of C. The given line integral may be written as

$$\begin{aligned}
\int_C -x^2y\,dx + xy^2\,dy &= \oint_{\partial D}\mathbf{F}\cdot d\mathbf{s} \\
&= \int\int_D\left(\frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(-x^2y)\right)\,dA \\
&= \int\int_D(y^2+x^2)\,dA,
\end{aligned}$$

   where $d\mathbf{s} = (dx, dy)$ and the second equality follows from Green's Theorem. It remains to evaluate the integral on the right-hand side of the above string of equalities. For this, swicth to polar coordinates and use $D = \{(r,\theta) : 0 \le r \le R, 0 \le \theta \le 2\pi\}$. We have

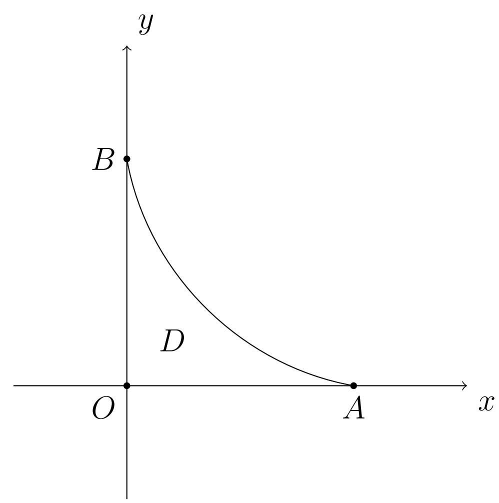
$$\int\int_D(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^R r^2\,r\,dr\,d\theta = \frac{1}{2}\pi R^4.$$

3. Verify that all hypotheses of Green's Theorem are satisfied.
   SolutionEach of the component functions of $\mathbf{F}$ has contionuous first-order partial derivatives on all of $\mathbb{R}^2$. Moreover ∂D is a circle, which is a smooth, simple closed curve.

Problem 5.28. Use a line integral to find the area $\mathcal{A}$ of the region D which is located in the first quadrant and is bounded by the curve

$$x^{1/2} + y^{1/2} = 1$$

and the segments OA and OB as shown in the figure.



SolutionThe area $\mathcal{A}$ of D is given by $\int\int_D \, dA$. To express this as a line integral, use Green's Theorem. That is, if $\mathbf{F} = (F_1, F_2)$ is any (sufficiently smooth) vector-valued function such that

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 \qquad \text{for } (x, y) \in D, \tag{5.17}$$

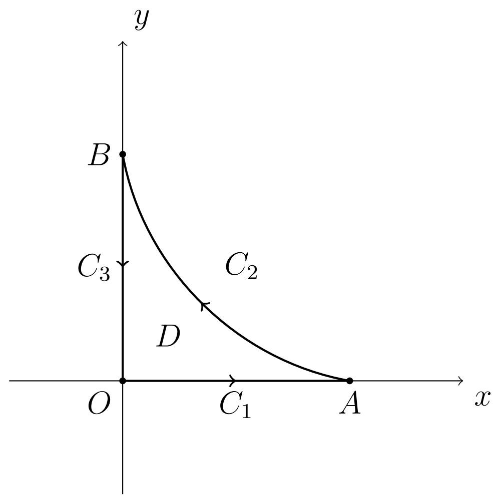
then

$$\mathcal{A} = \int\int_D \, dA = \int\int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}, \tag{5.18}$$

where ∂D is positively oriented. Clearly, the function $\mathbf{F}(x, y) = (0, x)$ satisfies (5.17), so by (5.18) we need only compute $\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ for this choice of $\mathbf{F}$. To achieve this, write

$$\partial D = C_1 + C_2 + C_3$$

where C_1 is the segment OA parameterized from O to A, C_2 is the curve $x^{1/2} + y^{1/2} = 1$ parameterized from A to B and C_3 is the segment OB parameterized from B to O (see the figure below).



Since

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1+C_2+C_3} \mathbf{F} \cdot d\mathbf{r} = \sum_{j=1}^{3} \int_{C_j} \mathbf{F} \cdot d\mathbf{r}, \tag{5.19}$$

we separately compute $\int_{C_j} \mathbf{F} \cdot d\mathbf{r}$ for $j = 1, 2, 3$.
The computation for $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ is as follows:
The function $\mathbf{r}_1(t) = (t, 0)$ for $0 \le t \le 1$ is a parameterization for C_1 with $\mathbf{r}_1'(t) = (1, 0)$. Moreover, $\mathbf{F}(\mathbf{r}_1(t)) = (0, t)$. Therefore, on C_1 we have $\mathbf{F} \cdot d\mathbf{r} = (0, t) \cdot (1, 0)\ dt = 0$, so

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0.$$

The computation of $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ is as follows:
The function $\mathbf{r}_2(t) = (t, (1 - \sqrt{t})^2)$ for $0 \le t \le 1$ is a parameterization for $-C_2$ (because this parameterization traverses C_2 from B to A). Moreover, $\mathbf{F}(\mathbf{r}_2(t)) = (0, t)$ and $\mathbf{r}_2'(t) = (1, 1 - 1/\sqrt{t})$. Therefore,

$$\mathbf{F} \cdot d\mathbf{r} = t\left(1 - \frac{1}{\sqrt{t}}\right)\ dt = (t - \sqrt{t})\ dt.$$

Finally,

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -\int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = -\int_0^1 (t - \sqrt{t})\ dt = \frac{1}{6}.$$

The computation for $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$ is as follows:
The function $\mathbf{r}_3(t) = (0, 1 - t)$ for $0 \le t \le 1$ is a parameterization for C_3. Moreover, $\mathbf{F}(\mathbf{r_3}(t)) = (0, 0)$, so $\mathbf{F} \cdot d\mathbf{r} = 0$ on C_3. Therefore,

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 0.$$

Using equations (5.18) and (5.19) and the results of the computations of $\int_{C_j} \mathbf{F}\cdot d\mathbf{r}$ for $j = 1, 2, 3$ yields

$$\mathcal{A} = \oint_{\partial D} \mathbf{F}\cdot d\mathbf{r} = 0 + \frac{1}{6} + 0 = \frac{1}{6}.$$

Problem 5.29. Use Stokes' Theorem to evaluate the circulation of the vector field $\mathbf{F} = \langle y, -x, z\rangle$ along the curve C given by the intersection of the cylinder $(x-1)^2 + y^2 = 1$ and the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$. Assume C is oriented counterclockwise as viewed from the top of the z-axis.

SolutionThe circulation of $\mathbf{F}$ along C is $\oint_C \mathbf{F}\cdot d\mathbf{r}$. Stokes' Theorem states that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int\int_S \nabla\times\mathbf{F}\cdot\widehat{\mathbf{n}}\, dS, \tag{5.20}$$

where S is the part of the hemisphere inside of the cylinder. That is, S is the graph of the function

$$g(x, y) = \sqrt{4 - x^2 - y^2} \qquad (x, y) \in D = \{(x, y) : (x-1)^2 + y^2 \leq 1\}.$$

Therefore, the function

$$\mathbf{r}(x, y) = (x, y, \sqrt{4 - x^2 - y^2}) \qquad (x, y) \in D$$

is a parameterization for S.
Now, routine computations show

$$\nabla\times\mathbf{F} = (0, 0, -2) \tag{5.21}$$

and

$$\widehat{\mathbf{n}} = \frac{\mathbf{r}_x\times\mathbf{r}_y}{\|\mathbf{r}_x\times\mathbf{r}_y\|} = \frac{(-g_x, -g_y, 1)}{\|\mathbf{r}_x\times\mathbf{r}_y\|}. \tag{5.22}$$

Moreover

$$dS = \|\mathbf{r}_x\times\mathbf{r}_y\|\ dA. \tag{5.23}$$

Using (5.21), (5.25) and (5.23) in (5.20) yields

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int\int_D (0, 0, 2)\cdot\frac{(-g_x, -g_y, 1)}{\cancel{\|\mathbf{r}_x\times\mathbf{r}_y\|}}\cancel{\|\mathbf{r}_x\times\mathbf{r}_y\|}\, dA \\
&= \int\int (-2)\, dA \\
&= -2\text{Area}(D) \\
&= -2\pi,
\end{aligned}$$

where the fact that D is a circle of radius 1 was used to deduce that $\text{Area}(D) = \pi$.

Problem 5.30. Consider the vector field $\mathbf{F} = \langle xy^2, x^2y, z\rangle$ and the surface S which is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$. Suppose S is oriented downward.

1. Compute the flux Φ of $\mathbf{F}$ across S via direct computation (i.e. use a surface integral involving $\mathbf{F}$).
   SolutionSince S is the graph of the function

$$g(x, y) = x^2 + y^2 \qquad (x, y) \in D = \{(x, y) : x^2 + y^2 \leq 1\},$$

a convenient parameterization for S is

$$\mathbf{r}(x, y) = (x, y, g(x, y)) \qquad (x, y) \in D. \tag{5.24}$$

Moreover, the downward unit normal vector on S is

$$\widehat{\mathbf{n}} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{\|\mathbf{r}_x \times \mathbf{r}_y\|} = \frac{(g_x, g_y, -1)}{\|\mathbf{r}_x \times \mathbf{r}_y\|} = \frac{(2x, 2y, -1)}{\|\mathbf{r}_x \times \mathbf{r}_y\|}. \tag{5.25}$$

Also, the area element dS for S is

$$dS = \|\mathbf{r}_x \times \mathbf{r}_y\| \; dA. \tag{5.26}$$

Finally, using equations (5.24), (5.25) and (5.26) the desired flux Φ is

$$\begin{aligned}
\Phi &= \int\!\!\int_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS \\
&= \int\!\!\int_S (2x^2y^2 + 2x^2y^2 - z)\frac{1}{\cancel{\|\mathbf{r}_x \times \mathbf{r}_y\|}}\cancel{\|\mathbf{r}_x \times \mathbf{r}_y\|}\, dA \\
&= \int\!\!\int_D (4x^2y^2 - x^2 - y^2)\, dA. \qquad (5.27)
\end{aligned}$$

To compute the integral on the right-hand side of equation (5.27), convert to polar coordinates and use each of the the double angle formulas $2\sin\alpha\cos\alpha = \sin(2\alpha)$ and $\sin^2\alpha = (1 - \cos(2\alpha))/2$ to get

$$\begin{aligned}
\int\!\!\int_D (r^4(2\sin\theta\cos\theta)^2 - r^2)\, dA &= \int\!\!\int_D (r^4\cos^2(2\theta) - r^2)\, dA \\
&= \int_0^1\int_0^{2\pi}\left(\frac{r^4}{2}(1 - \cos(4\theta)) - r^2\right) r\, d\theta\, dr \\
&= 2\pi\int_0^1\left(\frac{r^5}{2} - r^3\right) dr \\
&= -\frac{\pi}{3}.
\end{aligned}$$

2. Use the Divergence Theorem to compute the downward flux of $\mathbf{F}$ through S.
SolutionLet S_1 be the part of the plane $z = 1$ inside of the paraboloid $z = x^2 + y^2$. Then S and S_1 enclose a solid region which will be denoted E. Note that $\partial E = S \cup S_1$. Observe that the outward orientation of ∂E corresponds to the downward orientation for S. With such an orientation for ∂E, the Divergence Theorem yields

$$\int\int\int_E \nabla \cdot \mathbf{F}\, dV = \int\int_{\partial E} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS = \int\int_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS + \int\int_{S_1} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS, \tag{5.28}$$

where $\int\int_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$ is the downward flux of $\mathbf{F}$ through S. In view of (5.28), we only need to compute both

$$I_1 = \int\int\int_E \nabla \cdot \mathbf{F}\, dV \quad \text{and} \quad I_2 = \int\int_{S_1} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS. \tag{5.29}$$

The computation of I_1 (with minor details omitted) is as follows

$$\begin{aligned} I_1 &= \int\int\int_E (y^2 + x^2 + 1)\, dV \\ &= \int_0^1 \int_{r^2}^1 \int_0^{2\pi} (1 + r^2) r\, d\theta\, dz\, dr \\ &= \frac{2\pi}{3}. \end{aligned}$$

The computation of I_2 (with minor details omitted) is as follows

$$\begin{aligned} I_2 &= \int\int_{S_1} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS \\ &= \int\int_{S_1} (xy^2, x^2y, z) \cdot (0,0,1)\, dS \\ &= \int\int_{S_1} z\, dS \\ &= \int\int_{S_1} dS \\ &= \text{Area}(S_1) \\ &= \pi. \end{aligned}$$

Using the computations for I_1 and I_2 in equation (5.28) gives

$$\int\int_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS = -\frac{\pi}{3}.$$

Problem 5.31. Let S be the boundary of the smaller domain enclosed by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 2$. Use the Divergence Theorem to evaluate the outward flux of $\mathbf{F} = \langle x^2, y^2, z\rangle$ across S. Graph S and indicate the unit normal vector which corresponds to the orientation of S.

SolutionThe outward flux of $\mathbf{F}$ across S is $\int\int_S \mathbf{F}\cdot\widehat{\mathbf{n}}\, dS$, where $\widehat{\mathbf{n}}$ is the outward unit normal vector field on S. By the divergence theorem we have

$$\int\int_S \mathbf{F}\cdot\widehat{\mathbf{n}}\, dS = \int\int\int_E \nabla\cdot\mathbf{F}\, dV, \tag{5.30}$$

where E is the solid enclosed by S. We will compute the right-hand side of equation (5.30). Elementary computations yield

$$\nabla\cdot\mathbf{F} = 2x+2y+1,$$

so since $E = \{(\rho,\phi,\theta) : 0\le\rho\le\sqrt{2}, 0\le\phi\le\pi/4, 0\le\theta\le 2\pi\}$, we have

$$\begin{aligned}
\int\int\int_E \nabla\cdot\mathbf{F}\, dV &= \int\int\int_E (2x+2y+1)\, dV \\
&= \int_0^{\pi/4}\int_0^{\sqrt{2}}\int_0^{2\pi} (2\rho\sin\phi(\cos\theta+\sin\theta)+1)\,\rho^2\sin\phi\, d\theta\, d\rho\, d\phi \\
&= 2\pi\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^2\sin\phi\, d\rho\, d\phi \\
&= \frac{4\pi}{3}(\sqrt{2}-1).
\end{aligned}$$

Problem 5.32. If E is a solid region whose boundary is a closed surface S, the volume V of E may be represented by the surface integral

$$V = \frac{1}{3}\int\int_S \mathbf{r}\cdot\widehat{\mathbf{n}}\, dS.$$

Suppose S_1 is the surface parameterized by

$$\mathbf{r}(u,v) = \left\langle u\cos v, u\sin v, u^2\right\rangle \qquad 0\le u\le 1, 0\le v\le 2\pi$$

with orientation $\widehat{\mathbf{n}} = -\mathbf{r}_u\times\mathbf{r}_v/\,\|\mathbf{r}_u\times\mathbf{r}_v\|$. Suppose S_2 is the upward-oriented surface given by

$$S_2 = \{(x,y,z) : x^2+y^2\le 1, z=1\}.$$

If E is the solid whose boundary is the closed surface $S = S_1\cup S_2$, use the above surface integral to compute the volume V of E.

SolutionSince the given volume formula is

$$3V = \int\int_S \mathbf{r}\cdot\widehat{\mathbf{n}}\,dS = \int\int_{S_1} \mathbf{r}\cdot\widehat{\mathbf{n}}\,dS + \int\int_{S_2} \mathbf{r}\cdot\widehat{\mathbf{n}}\,dS, \tag{5.31}$$

we wish to compute each of

$$I_1 = \int\int_{S_1} \mathbf{r}\cdot\widehat{\mathbf{n}}\,dS \quad \text{and} \quad I_2 = \int\int_{S_2} \mathbf{r}\cdot\widehat{\mathbf{n}}\,dS.$$

To compute I_1, use the given parameterization of S_1 and perform routine computations to get

$$-\mathbf{r}_u \times \mathbf{r}_v = (2u^2\cos v, 2u^2\sin v, u)$$

and

$$-\mathbf{r}\cdot(\mathbf{r}_u \times \mathbf{r}_v) = u^3.$$

Therefore

$$\begin{aligned} I_1 &= \int\int_{S_1} \mathbf{r}\cdot\widehat{\mathbf{n}}\,dS \\ &= \int_0^{2\pi}\int_0^1 -\mathbf{r}\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{\cancel{\|\mathbf{r}_u\times\mathbf{r}_v\|}}\cancel{\|\mathbf{r}_u\times\mathbf{r}_v\|}\,du\,dv \\ &= \int_0^{2\pi}\int_0^1 u^3\,du\,dv \\ &= \frac{\pi}{2}. \end{aligned} \tag{5.32}$$

To compute I_2 observe that $\widehat{\mathbf{n}} = (0,0,1)$ so, since $z = 1$ on S_2,

$$\mathbf{r}\cdot\widehat{\mathbf{n}} = z = 1.$$

Therefore

$$\begin{aligned} I_2 &= \int\int_{S_2} \mathbf{r}\cdot\widehat{\mathbf{n}}\,dS \\ &= \int\int_{S_2} 1\,dS \\ &= \text{Area}(S_2) \\ &= \pi. \end{aligned} \tag{5.33}$$

Using equations (5.32) and (5.33) in (5.31) yields

$$V = \frac{1}{3}\left(\frac{\pi}{2} + \pi\right) = \frac{\pi}{2}.$$