Counting Patterns
Equipopularity in Permutation Classes

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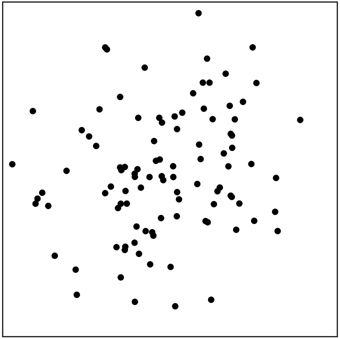
Introduction

Random Data



Permutations

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Definition

An *permutation of length n* is a bijection from the set $[n] = \{1, 2, \ldots n\}$ to itself. The *one-line notation* for a permutation π is

$$\pi = \pi(1)\pi(2)\ldots\pi(n).$$

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Examples

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Examples

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- The six permutations of length 3 are

$$\mathfrak{S}_3 = \{123, 132, 213, 231, 312, 321\}.$$

Plotting Permutations

Definition

If π is a permutation of length *n*, then the *plot* of π is the set of points

$$\{(1, \pi(1)), (2, \pi(2)), \cdots (n, \pi(n))\} \subset \mathbb{R}^2$$

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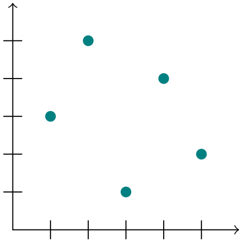

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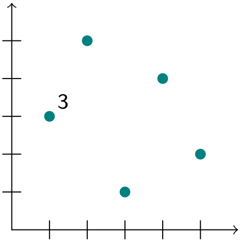

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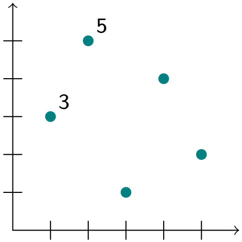

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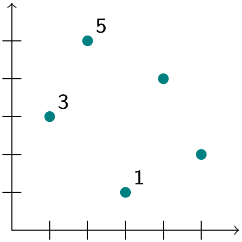

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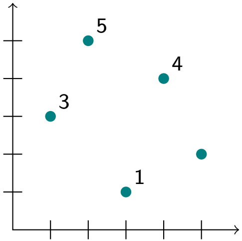

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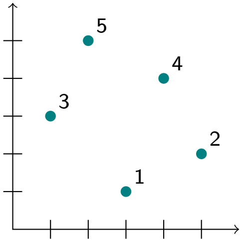

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Dots on a Plane

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Let A and B be two sets of n points in $\mathbb{R}^2$, each with the property that no two points lie on the same horizontal or vertical line.
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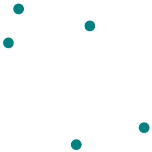

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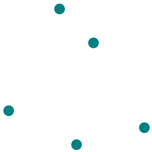

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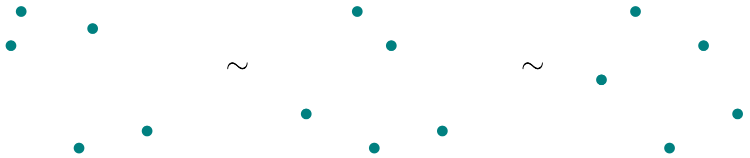

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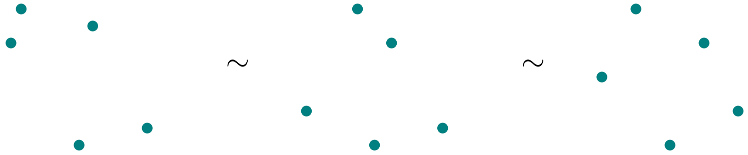

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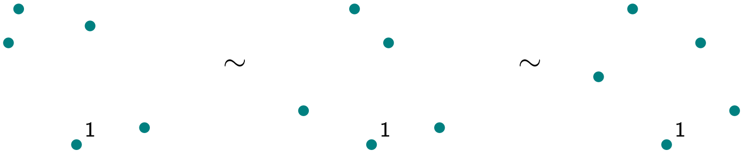

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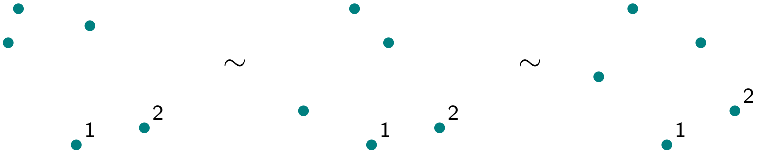

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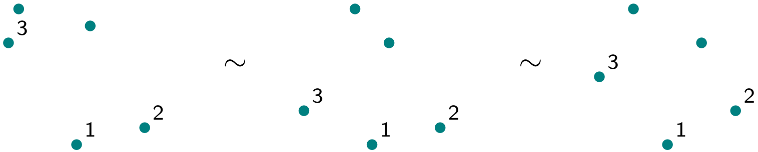

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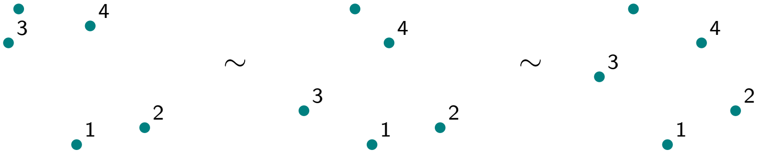

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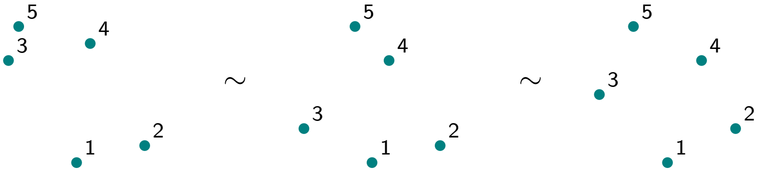

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Permutation Symmetries

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For a permutation $\pi = \pi_1\pi_2\ldots\pi_n$, the reverse, the complement, and the inverse of π are denoted π^r, π^c, and π^{-1}, and defined as follows:

$$(\pi^r)_i = \pi_{n+1-i}, \quad (\pi^c)_i = n+1-\pi_i, \text{ and}$$

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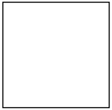

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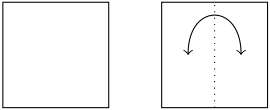

π $\quad$ π^r

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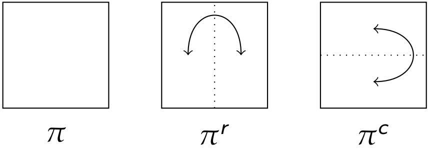

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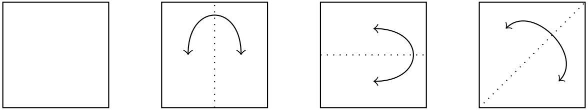

π π^r π^c π^{-1}

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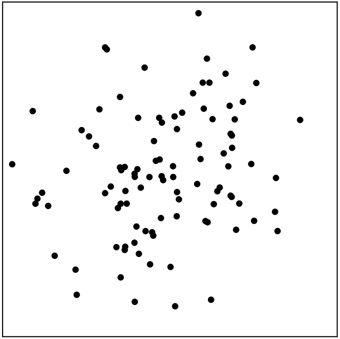

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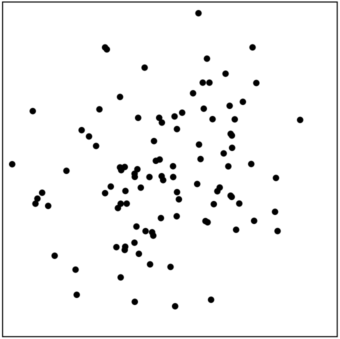

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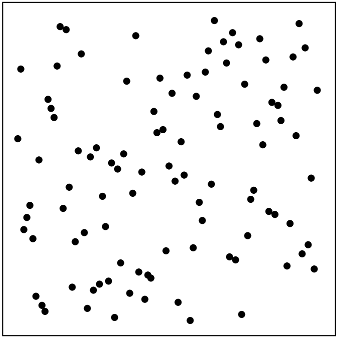

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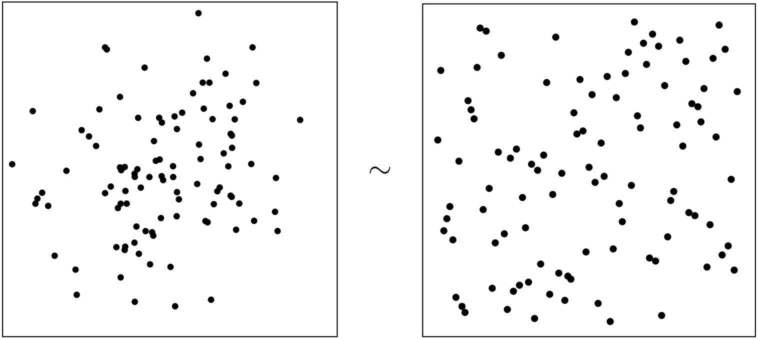

$\pi =$ 61 84 31 35 39 28 9 54 6 4 74 71 68 85 98 38 97 45 12 27 57 89 30 5 55 11 58
13 42 32 14 53 2 51 20 56 80 10 43 95 17 50 8 16 15 70 63 81 64 24 52 76 47
7 60 49 82 1 25 75 40 34 83 90 46 100 69 65 93 86 22 96 21 92 3 79 29 41
44 66 94 59 87 37 73 36 72 67 78 19 33 88 62 99 23 91 26 48 18 77

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Let $\pi = \pi(1)\pi(2)\cdots\pi(n)$ and $\sigma = \sigma(1)\sigma(2)\cdots\sigma(k)$ be two permutations. π *contains* σ *as a pattern* (written $\sigma \prec \pi$) if there is some subsequence $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ which is order isomorphic to the entries of σ (i.e., $\pi(i_j) < \pi(i_k)$ if and only if $\sigma(j) < \sigma(k)$).

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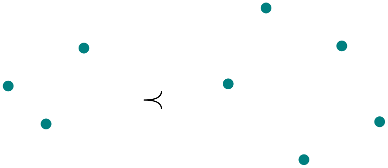

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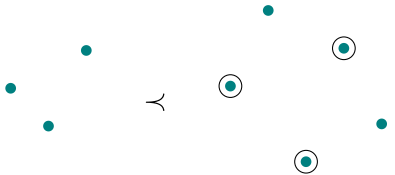

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Example

The pattern 12 is contained in all permutations *except* for the decreasing ones:

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Definition

If a permutation π does not contain a pattern σ, we say that π *avoids* σ. The set of all permutations which avoid a given pattern (or set of patterns) σ is denoted

$$\mathrm{Av}(\sigma).$$

Permutation Classes

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A *permutation class* is a set $\mathcal{C}$ of permutations for which, if $\pi \in \mathcal{C}$ and $\sigma \prec \pi$, then $\sigma \in \mathcal{C}$. Let $\mathcal{C}_n$ denote the set of permutations of length n in $\mathcal{C}$.

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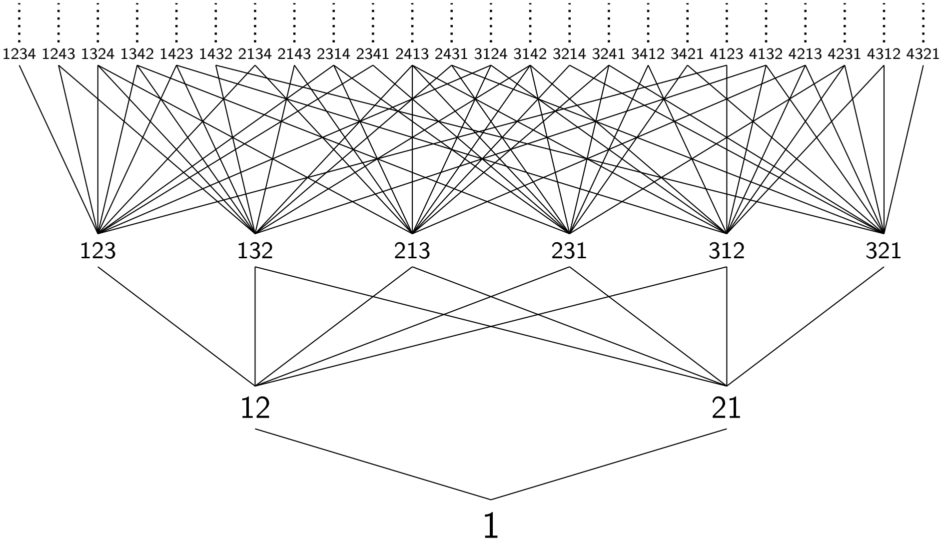
Theorem (Marcus and Tardos, 2004)

Every proper permutation class has a finite exponential growth rate. That is, for any proper class $\mathcal{C}$, there exists a real number s such that

$$\limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|} = s.$$

This number s is the *growth rate* of the class.

Permutation Classes - Growth Rates



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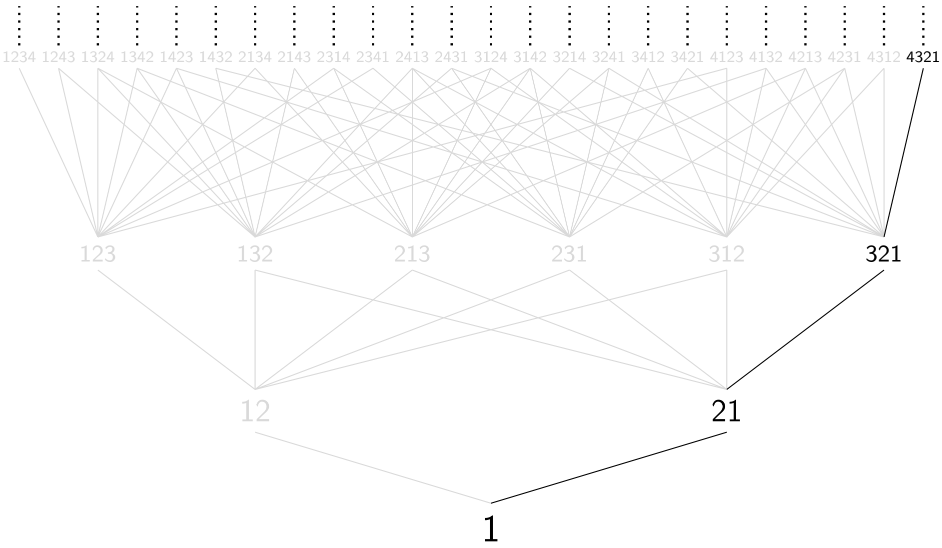

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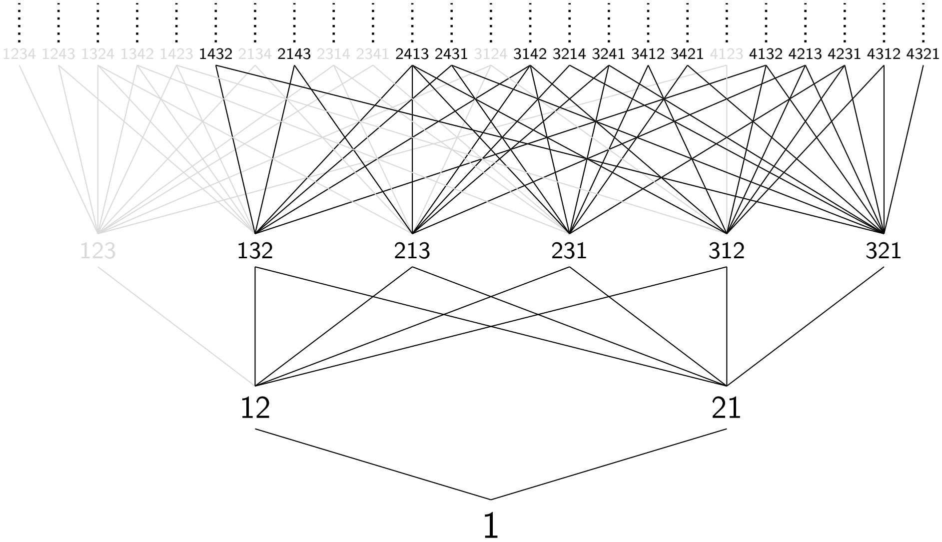

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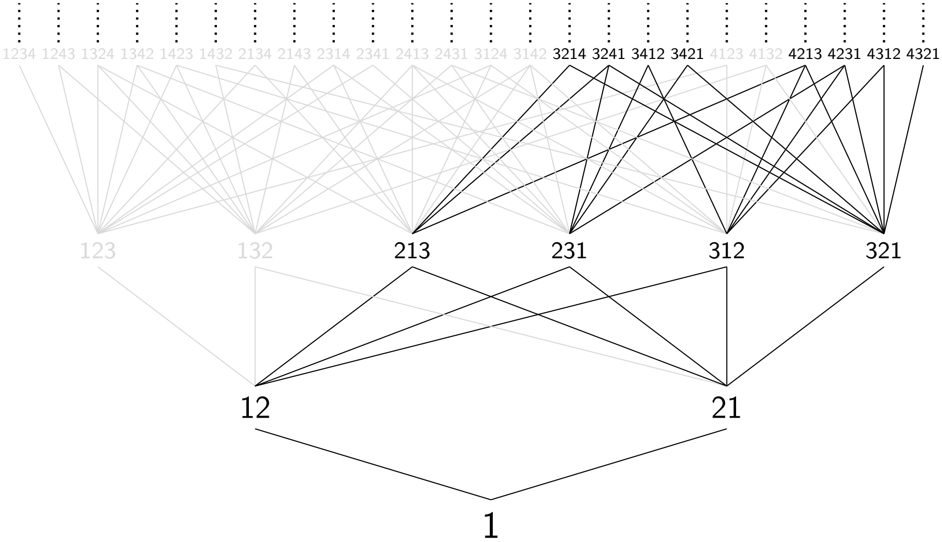

The Class Av(132)

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Let c_n be the number of permutations of length n which *avoid* the pattern 132, and $C(x) = \sum_{n \geq 0} c_n x^n$.

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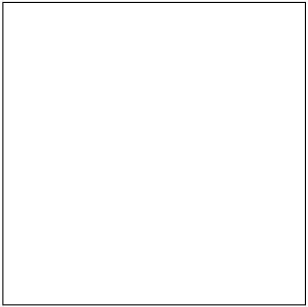

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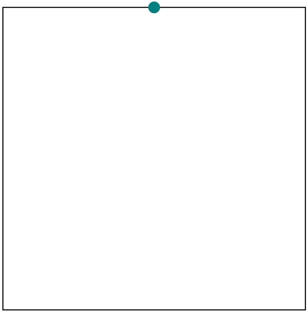

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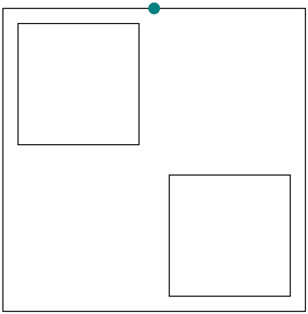

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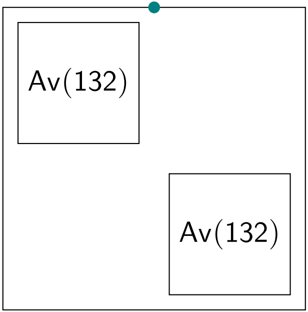

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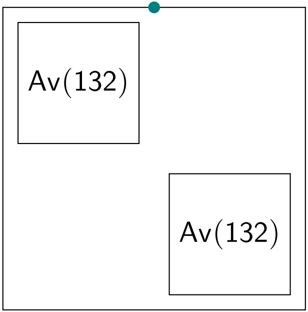

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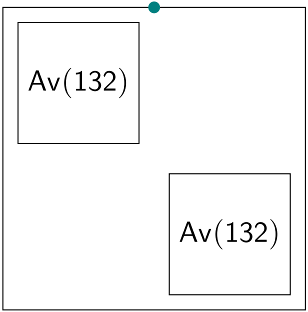

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$$0 = xC(x)^2 - C(x) + 1$$

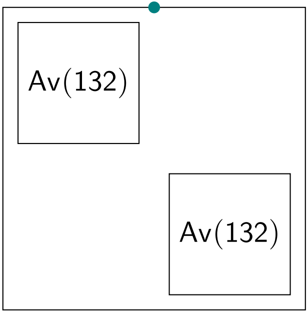
The Class Av(132)

Definition

Let c_n be the number of permutations of length n which *avoid* the pattern 132, and $C(x) = \sum_{n \geq 0} c_n x^n$.

Question

What does a 132-avoiding permutation look like?



$$C(x) = xC(x)^2 + 1$$

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$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

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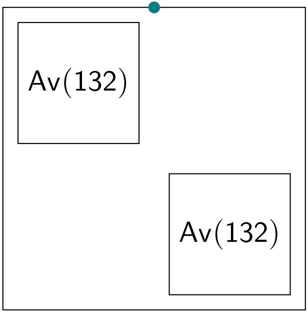

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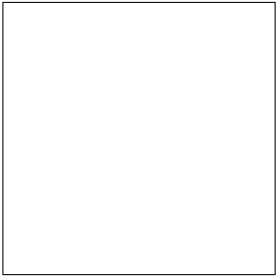

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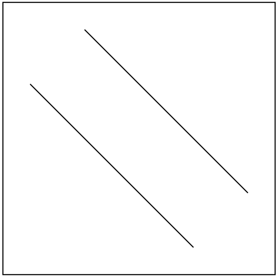

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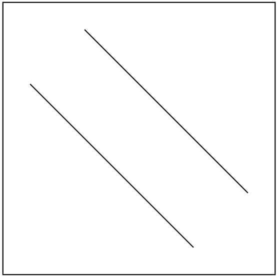

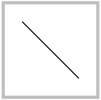

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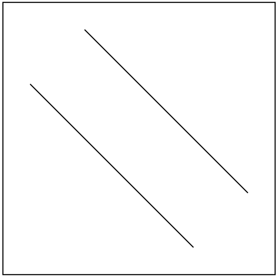

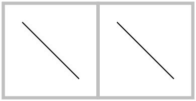

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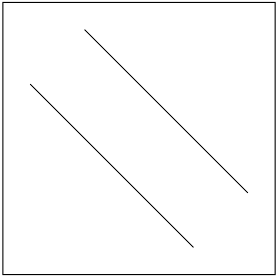

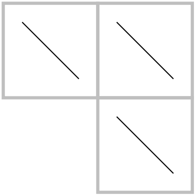

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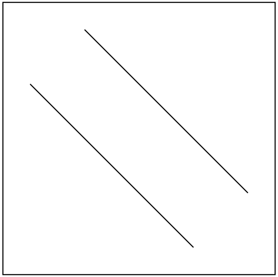

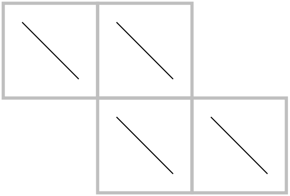

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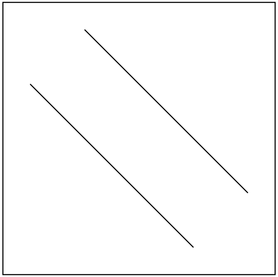

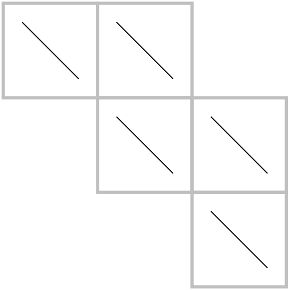

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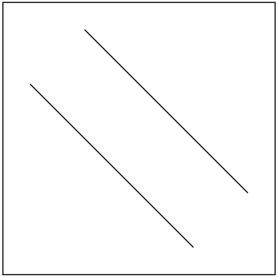

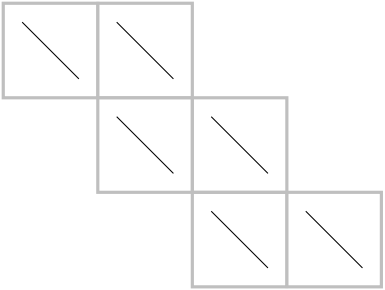

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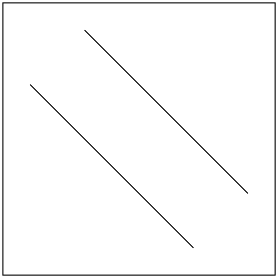
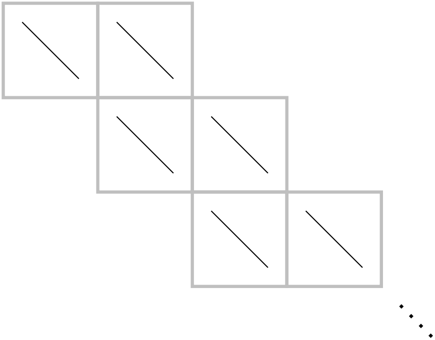
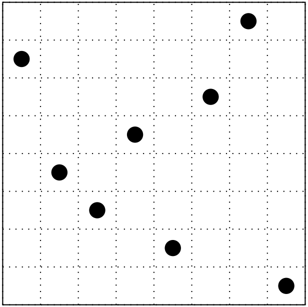




Av(132) and Av(123)



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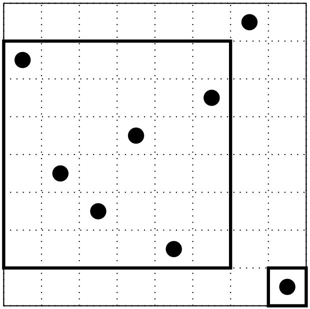

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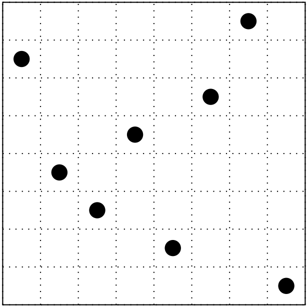

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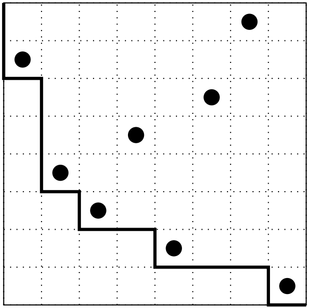

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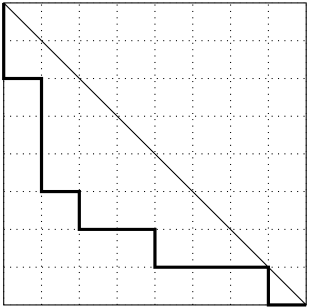

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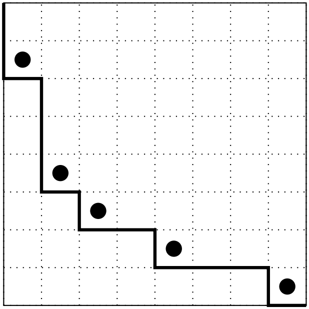

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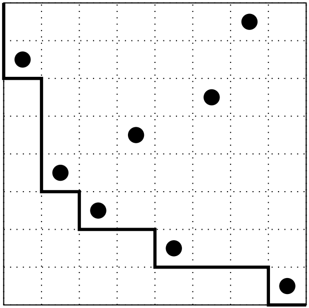

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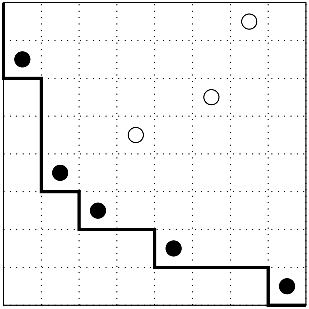

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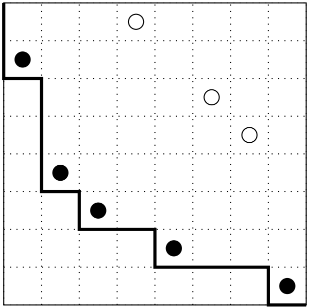
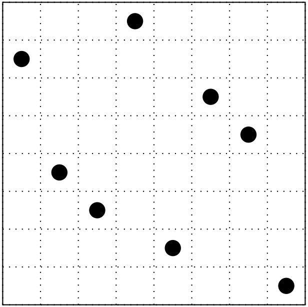


$\mathsf{Av}(132) \mapsto \mathsf{Av}(123)$

Av(132) and Av(123)



Av(123)

Av(132) and Av(123)

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$$|\operatorname{Av}_n(1324) = ???|$$

Pattern Occurrences

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Say that one permutation π contains another permutation σ *as a pattern* (denoted $\sigma \prec \pi$) if the plot of π contains a subset which is equivalent to the plot of σ. The number of occurrences of σ in π (denoted $\nu_\sigma(\pi)$) is the number of such subsets.

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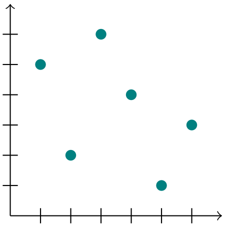

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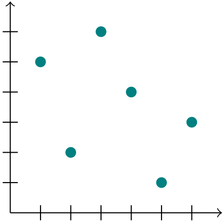

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$$\nu_{132}(526413) = 3$$

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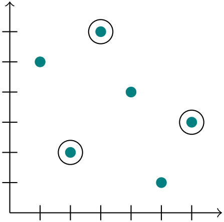

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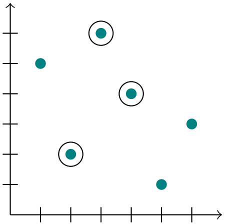

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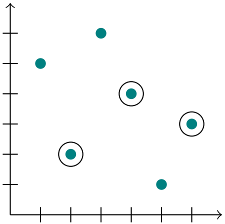

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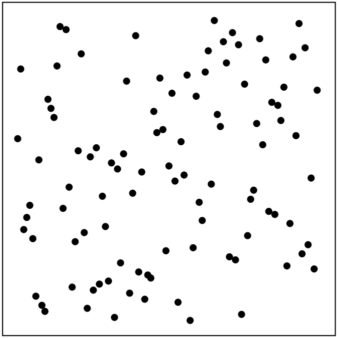
Random Data



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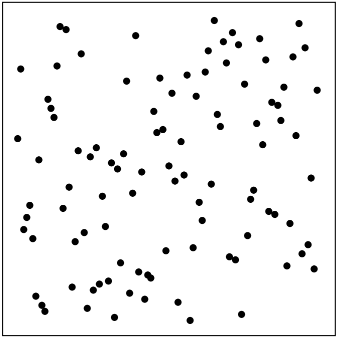

ν_{12}	ν_{21}	Avg
2803	2147	2475

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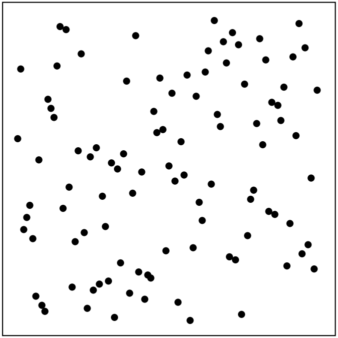

ν_{12}	ν_{21}	Avg
2803	2147	2475

ν_{123}	ν_{132}	ν_{213}	ν_{231}	ν_{312}	ν_{321}	Avg
35357	30063	31414	22321	23348	19197	26950

Patterns as Random Variables

Theorem (Bóna 2007)

For a (uniformly) randomly selected permutation of length n, the random variables ν_σ are asymptotically normal as n approaches infinity.

Theorem (Janson, Nakamura, Zeilberger 2013)

For a randomly selected permutation of length n and two patterns σ and ρ, the random variables ν_σ and ν_ρ are asymptotically jointly normally distributed as $n \to \infty$.

Motivation

Fact

In $\mathfrak{S}_n$, the number of occurrences of a specific pattern depends only on the length of the pattern. That is, for a pattern $\sigma \in \mathfrak{S}_k$, we have

$$\nu_\sigma(\mathfrak{S}_n) = \frac{n!}{k!}\binom{n}{k}.$$

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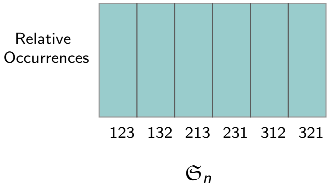

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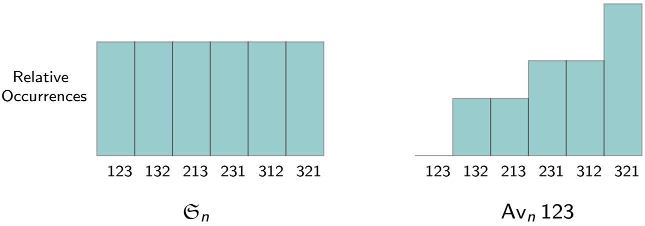

Connections Between Classes

Av(123) and Av(132)

Previous Results

Theorem (Bóna 2010)

In $\mathrm{Av}_n 132$, the pattern 123 is the least common, 321 is the most common, and $\nu_{213} = \nu_{231} = \nu_{312}$.

Data

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
4	10	0	11	11	11	13
5	68	0	81	81	81	109
6	392	0	500	500	500	748
7	2063	0	2794	2794	2794	4570

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
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Av 123

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3	0	1	1	1	1	1

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Av 123

length	123	132	213	231	312	321
3	0	1	1	1	1	1
4	0	9	9	11	11	16

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
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Av 123

length	123	132	213	231	312	321
3	0	1	1	1	1	1
4	0	9	9	11	11	16
5	0	57	57	81	81	144
6	0	312	312	500	500	1016
7	0	1578	1578	2794	2794	6271

Patterns Within Av(123)

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Theorem (H 2012)

The total nuber of 231 (and 312) patterns is identical within the sets $\mathrm{Av}_n(123)$ and $\mathrm{Av}_n(132)$.

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Theorem (H 2012)

The total nuber of 231 (and 312) patterns is identical within the sets $\mathrm{Av}_n(123)$ and $\mathrm{Av}_n(132)$.
Further, within $\mathrm{Av}_n(123)$,

$$\nu_{132} = \nu_{213} \sim \sqrt{\frac{n}{\pi}}4^n,$$
$$\nu_{231} = \nu_{312} \sim \frac{n}{2}4^n,$$
$$\text{and} \quad \nu_{321} \sim \frac{8}{3}\sqrt{\frac{n^3}{\pi}}4^n.$$

Sketch of Proof: Patterns in Av(123)

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ν_{132} ν_{213} ν_{231} ν_{312} ν_{321}

Sketch of Proof: Patterns in Av(123)

$$\nu_{132} + \nu_{213} + \nu_{231} + \nu_{312} + \nu_{321} = \binom{n}{3} c_n$$

(Both sides count the number of length three patterns)

Sketch of Proof: Patterns in Av(123)

$$2\nu_{132} + 2\nu_{213} + \nu_{231} + \nu_{312} = (n-2)\nu_{12}$$

(Count triples containing a 12 pattern ...)

Sketch of Proof: Patterns in Av(123)

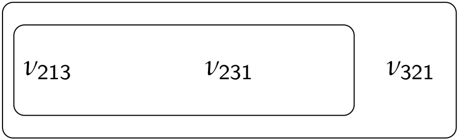
ν_{132} ν_{213} ν_{231} ν_{312} ν_{321}

Sketch of Proof: Patterns in Av(123)

$$\nu_{132} = \nu_{213} \qquad \nu_{231} = \nu_{312} \qquad \nu_{321}$$

(Since Av(123) is closed under inversion)

Sketch of Proof: Patterns in Av(123)



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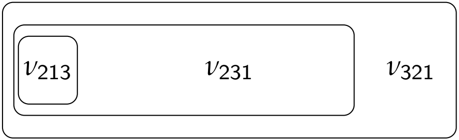

Sketch of Proof: Counting 213 Patterns

Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.

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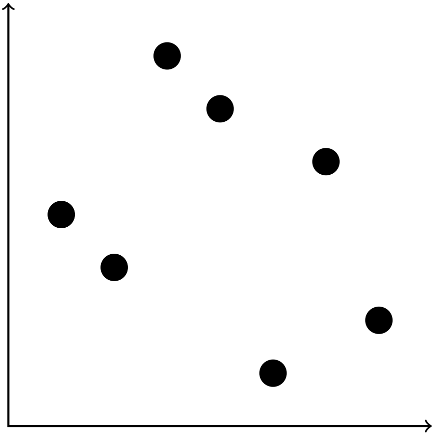

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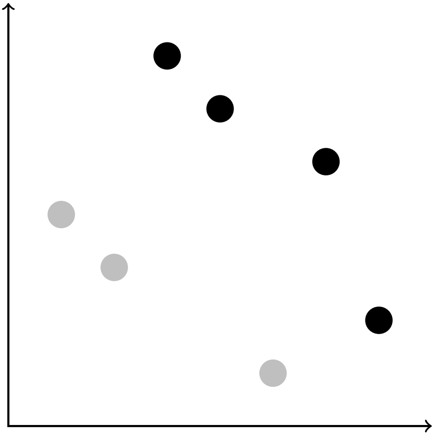

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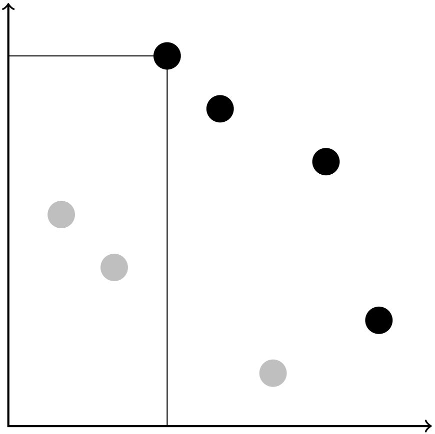

$$\nu_{213}(p) = \binom{2}{2}$$

Sketch of Proof: Counting 213 Patterns

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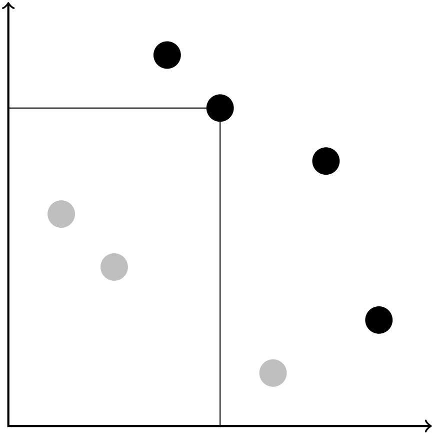
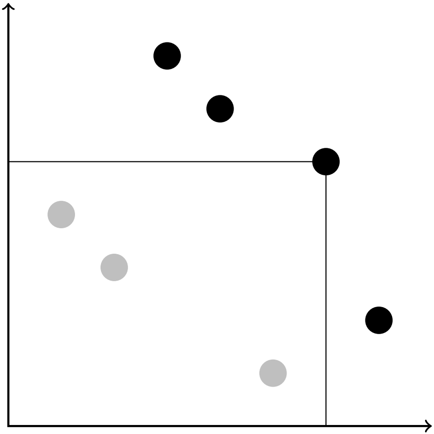


$$\nu_{213}(p) = \binom{2}{2} + \binom{2}{2}$$

Sketch of Proof: Counting 213 Patterns

Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.



$$\nu_{213}(p) = \binom{2}{2} + \binom{2}{2} + \binom{3}{2}$$

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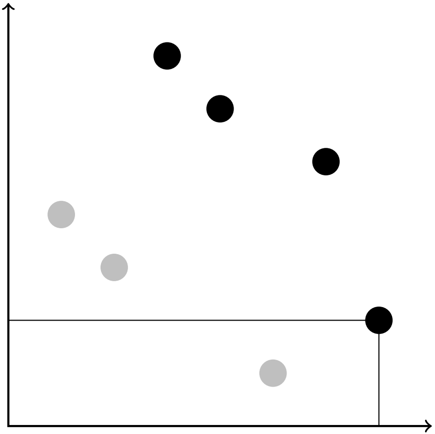

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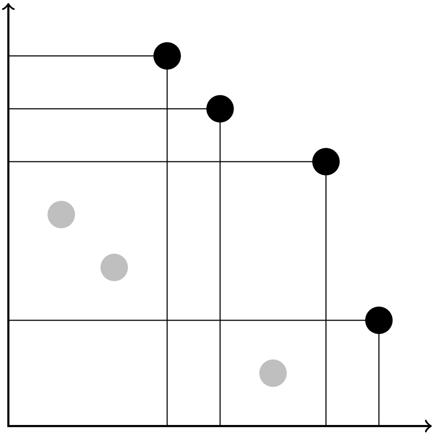

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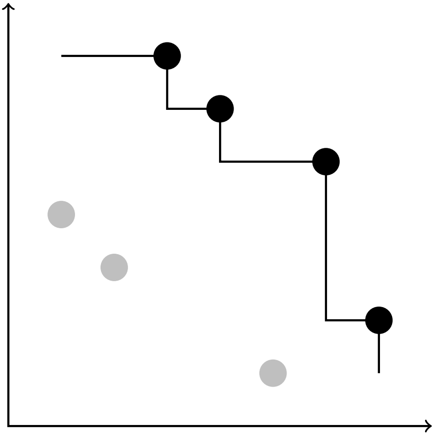

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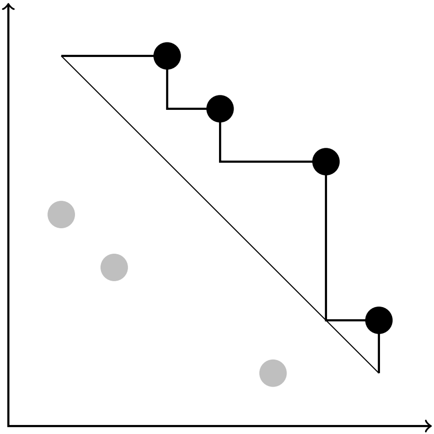

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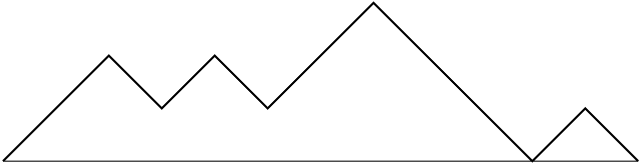

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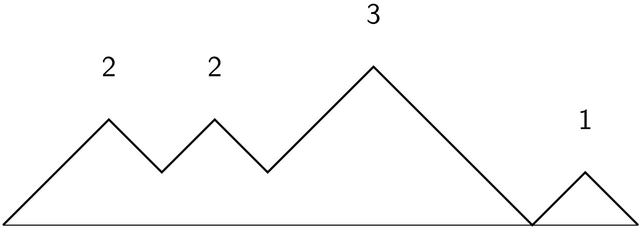

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Sketch of Proof: Counting 213 Patterns

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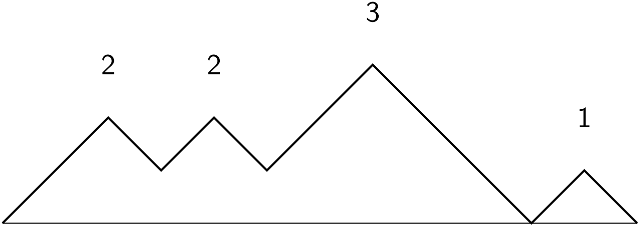

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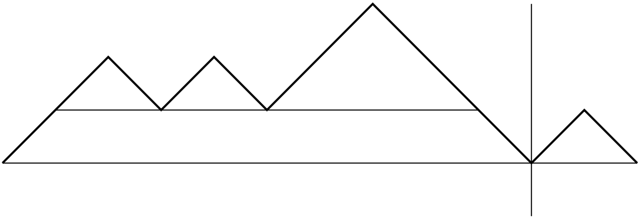

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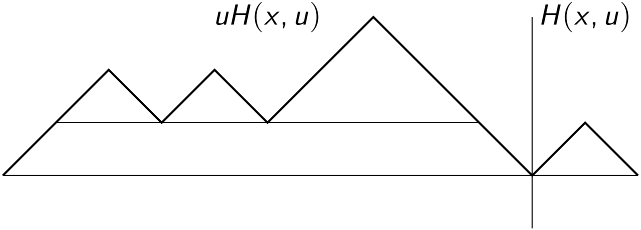

$$H(x,u) = ux(H(x,u)+1)C(x) + xC(x)H(x,u)$$

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□

Results

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$$\nu_{231}(\mathsf{Av}_n\, 123) = \nu_{231}(\mathsf{Av}_n\, 132)$$

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$$\nu_{213} = \frac{n+2}{4}\binom{2n}{n} - 3 \cdot 2^{2n-3}$$

$$\nu_{231} = (2n-1)\binom{2n-3}{n-2} - (2n+1)\binom{2n-1}{n-1} + (n+4) \cdot 2^{2n-3}$$

$$\begin{aligned}\nu_{321} = {} & \frac{1}{6}\binom{2n+5}{n+1}\binom{n+4}{2} - \frac{5}{3}\binom{2n+3}{n}\binom{n+3}{2} \\ & + \frac{17}{3}\binom{2n+1}{n-1}\binom{n+2}{2} - 6\binom{2n-1}{n-2}\binom{n+1}{2} - (n+1) \cdot 4^{n-1}.\end{aligned}$$

Connections Within Classes

Av(132) and the Separables

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Theorem (Chua, Sankar 2013)

If two patterns are equipopular in Av(132), then they *have the same structure*.

Corollary

The equipopularity classes within Av(132) are in bijection with the set of integer partitions.

Separable Permutations

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Definition

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Theorem (Albert, H, Pantone 2014)

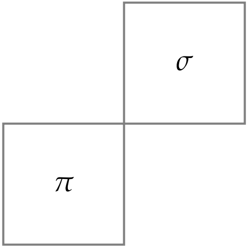
Two patterns are equipopular in the separables if and only if they *have the same structure*.

Separable Permutations

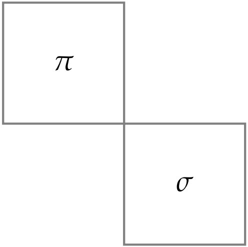
Separable Permutations

Definition

Given two permutations π and σ, their *direct sum* $(\pi \oplus \sigma)$ and *skew sum* $(\pi \ominus \sigma)$ are defined as follows:



$\pi \oplus \sigma$



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Alternate Definition

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Example

The permutation $\pi = 215643798$ is separable, since

$$\pi = \Big(1 \ominus 1\Big) \oplus \Big((1 \oplus 1) \ominus 1 \ominus 1\Big) \oplus 1 \oplus \Big(1 \ominus 1\Big).$$

Separable Permutations

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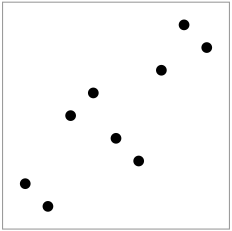

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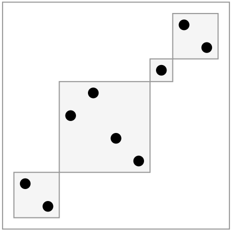

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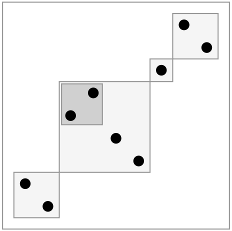

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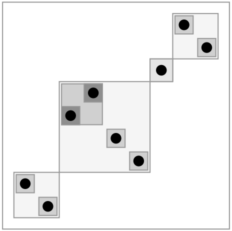

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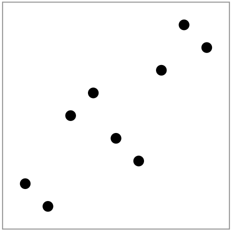

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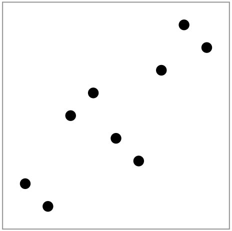

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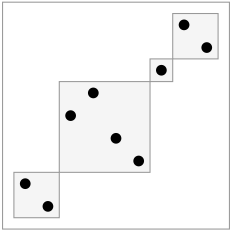

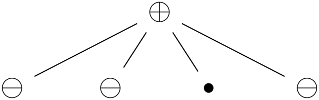

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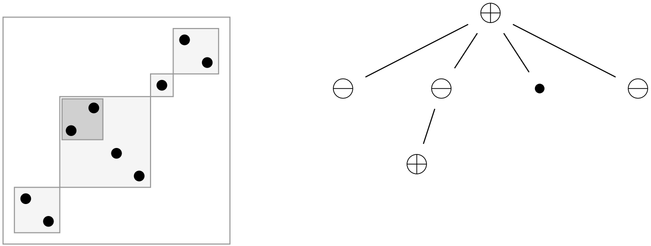

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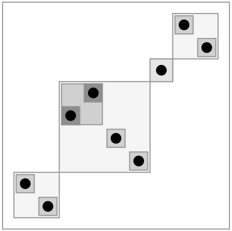

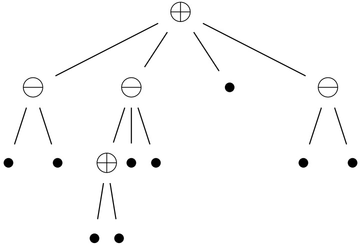

Tree Containment

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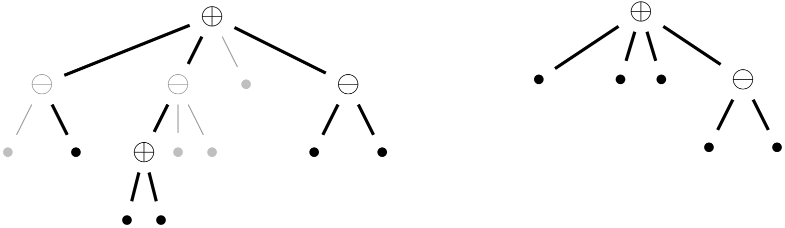

Equipopularity

Question

If two patterns are equipopular, how are their trees related?

Strategy

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Part 1

Find the operations on trees which preserve popularity.

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Find the operations on trees which preserve popularity.

Part 2

Show that equipopularity implies that their trees are related by one of these operations.

Preserving Popularity

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Symmetries

Permutation	Tree

Preserving Popularity

Symmetries

Permutation	Tree
Complementation	Flip signs

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Fact

If two permutations (trees) are related by any of the above symmetries, then they are equipopular.

Preserving Popularity - Shuffling

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Lemma

Rearranging the children of any node preserves equipopularity.

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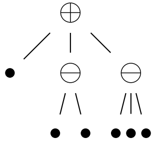

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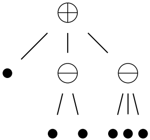

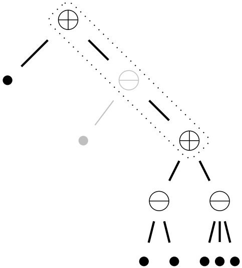

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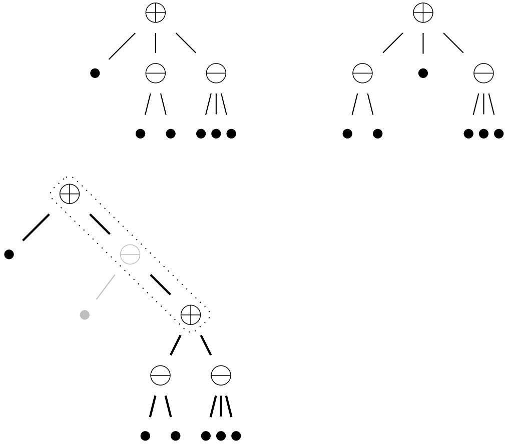

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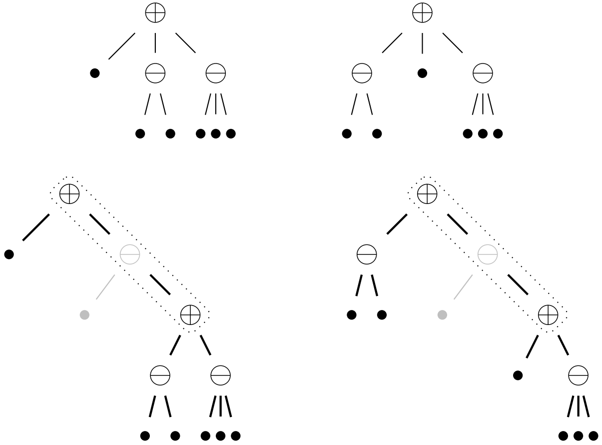
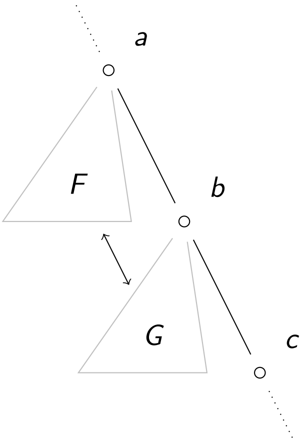


Preserving Popularity - Rotation



Preserving Popularity

Theorem (Albert, H, Pantone 2014)

The following operations preserve popularity:

- Reversal
- Complementation
- Inversion
- Shuffling
- Rotation

Canonical Representatives

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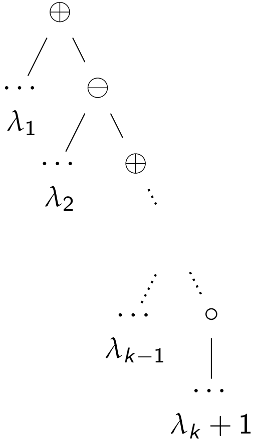
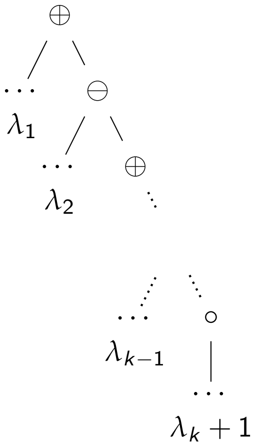


Canonical Representatives



$$\lambda := \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

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If two patterns are equipopular, one can be transformed into the other by the above operations.

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Corollary

The set of equipopularity classes for patterns of length n are in bijection with the set of partitions of the integer $n-1$.

Rough Sketch of Proof

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Idea # 1

Given any arbitrary pattern, we can factor its popularity generating function into the popularity generating functions for monotone runs.

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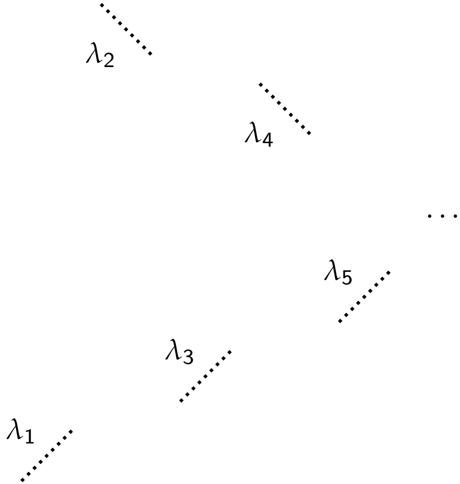

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How?

- Recursively build a bivariate popularity generating function for all monotone patterns.
- Notice (or let Sage tell you) that these are related to the *Gegenbauer polynomials*, a family of orthogonal polynomials.

Rough Sketch of Proof

Idea #2

Given a product of monotone popularity generating functions, we can uniquely factor into its component parts, and thus recover the lengths of each monotone pattern.

How?

- Recursively build a bivariate popularity generating function for all monotone patterns.
- Notice (or let Sage tell you) that these are related to the *Gegenbauer polynomials*, a family of orthogonal polynomials.
- Use the orthogonality of these polynomials to uniquely factor any product.

Future Work

What else?

What else?

Similar results within other classes?

What else?

Similar results within other classes?
Similar connections between classes?

What else?

Similar results within other classes?
Similar connections between classes?
Beyond averages: equidistribution?

Thank you for listening!