Pattern Popularity and Separable Permutations

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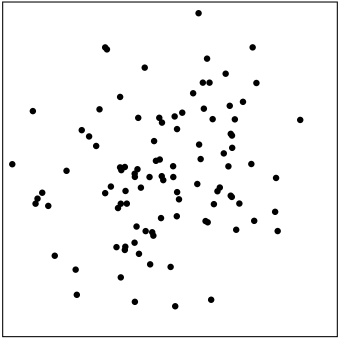
Howard University

September 12th, 2014

Introduction

Random Data



Permutations

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An *permutation of length n* is a bijection from the set $[n] = \{1, 2, \dots n\}$ to itself. The *one-line notation* for a permutation π is

$$\pi = \pi(1)\pi(2)\dots\pi(n).$$

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Examples

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Examples

- The sequence $\pi = 5172643$ is a permutation of length 7.
- The six permutations of length 3 are

$$\mathfrak{S}_3 = \{123, 132, 213, 231, 312, 321\}.$$

Plotting Permutations

Definition

If π is a permutation of length n, then the *plot* of π is the set of points

$$\{(1, \pi(1)), (2, \pi(2)), \cdots (n, \pi(n))\} \subset \mathbb{R}^2$$

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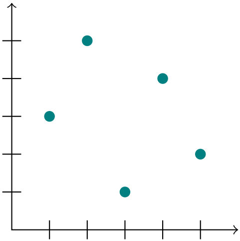

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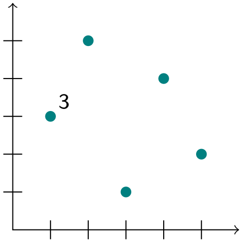

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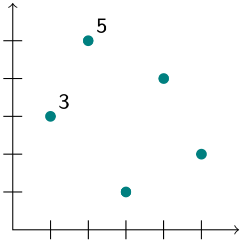

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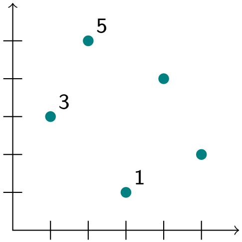

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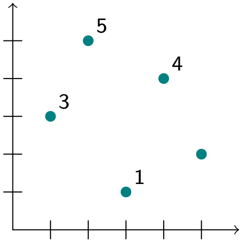

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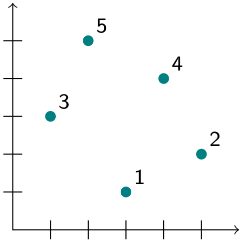

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Let A and B be two sets of n points in $\mathbb{R}^2$, each with the property that no two points lie on the same horizontal or vertical line.
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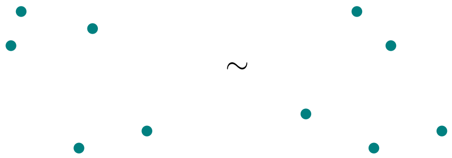

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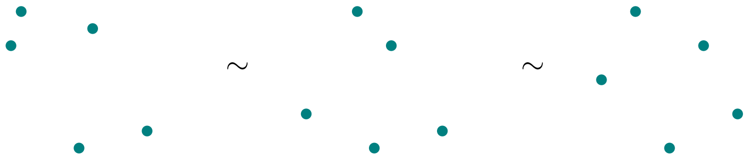

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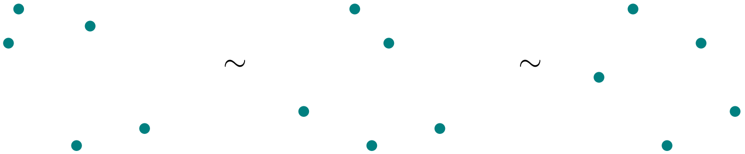

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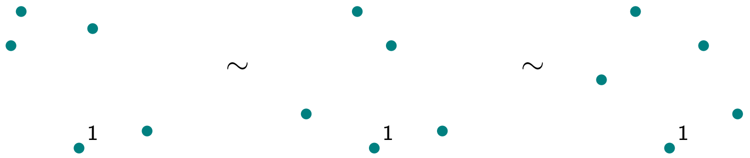

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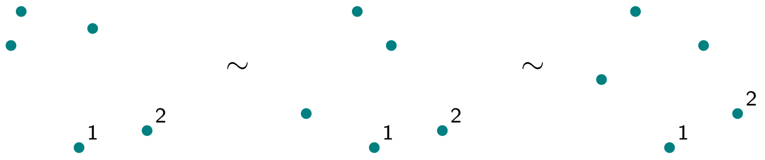

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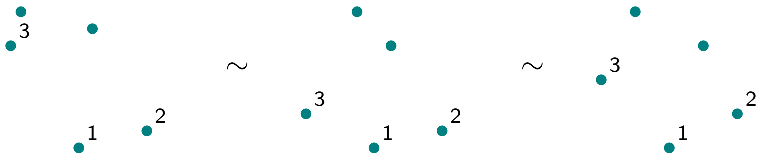

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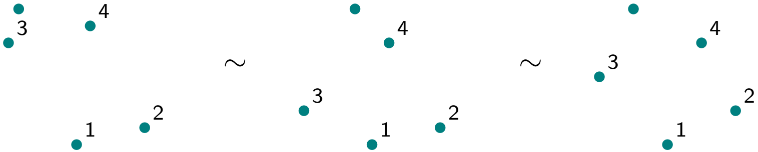

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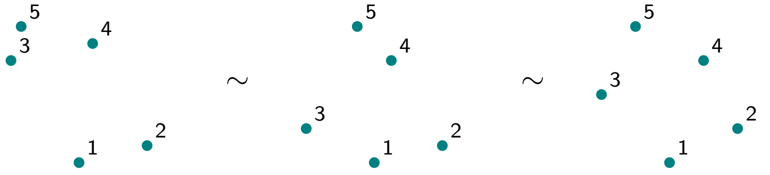

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For a permutation $\pi = \pi_1\pi_2 \ldots \pi_n$, the reverse, the complement, and the inverse of π are denoted π^r, π^c, and π^{-1}, and defined as follows:

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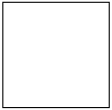

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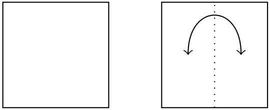

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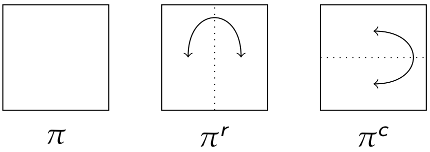

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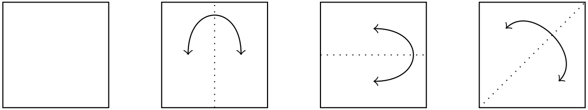

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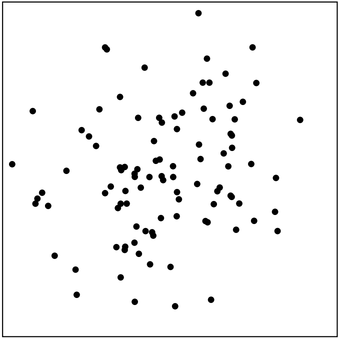

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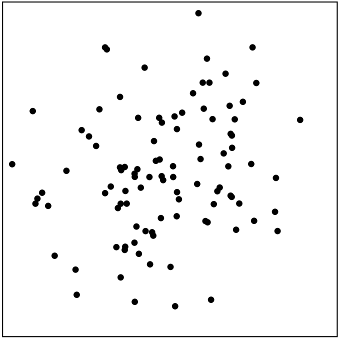
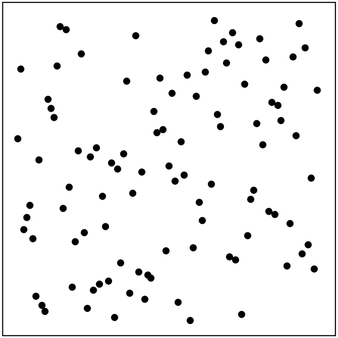


$\sim$



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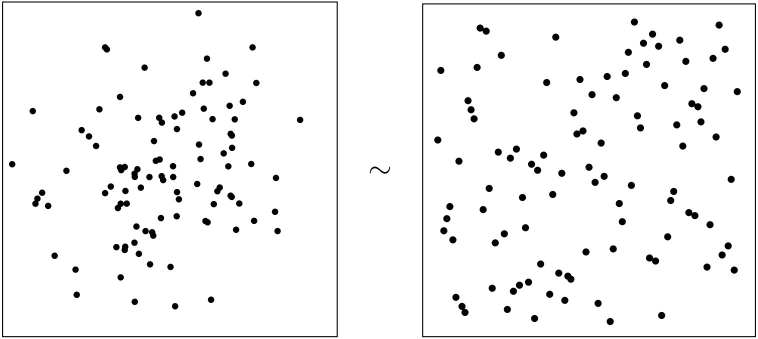

$\pi =$ 61 84 31 35 39 28 9 54 6 4 74 71 68 85 98 38 97 45 12 27 57 89 30 5 55 11 58
13 42 32 14 53 2 51 20 56 80 10 43 95 17 50 8 16 15 70 63 81 64 24 52 76 47
7 60 49 82 1 25 75 40 34 83 90 46 100 69 65 93 86 22 96 21 92 3 79 29 41
44 66 94 59 87 37 73 36 72 67 78 19 33 88 62 99 23 91 26 48 18 77

Permutation Patterns

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Let $\pi = \pi(1)\pi(2)\cdots\pi(n)$ and $\sigma = \sigma(1)\sigma(2)\cdots\sigma(k)$ be two permutations. π *contains* σ *as a pattern* (written $\sigma \prec \pi$) if there is some subsequence $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ which is order isomorphic to the entries of σ (i.e., $\pi(i_j) < \pi(i_k)$ if and only if $\sigma(j) < \sigma(k)$).

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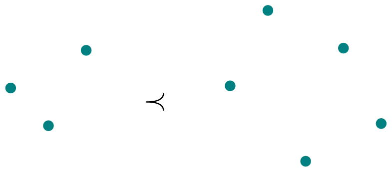

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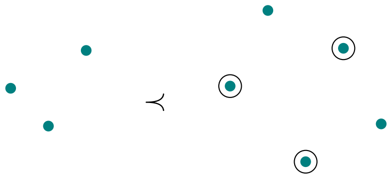

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Permutation Patters

Example

The pattern 12 is contained in all permutations *except* for the decreasing ones:

$$12 \not\prec n \ldots 321.$$

Example

The pattern 12 is contained in all permutations *except* for the decreasing ones:

$$12 \not\leq n \ldots 321.$$

Definition

If a permutation π does not contain a pattern σ, we say that π *avoids* σ. The set of all permutations which avoid a given pattern (or set of patterns) σ is denoted

$$\mathrm{Av}(\sigma).$$

Permutation Classes

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A *permutation class* is a set $\mathcal{C}$ of permutations for which, if $\pi \in \mathcal{C}$ and $\sigma \prec \pi$, then $\sigma \in \mathcal{C}$. Let $\mathcal{C}_n$ denote the set of permutations of length n in $\mathcal{C}$.

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The *enumeration* of a class $\mathcal{C}$ is the sequence $|\mathcal{C}_1|, |\mathcal{C}_2|, |\mathcal{C}_3|, \ldots$.

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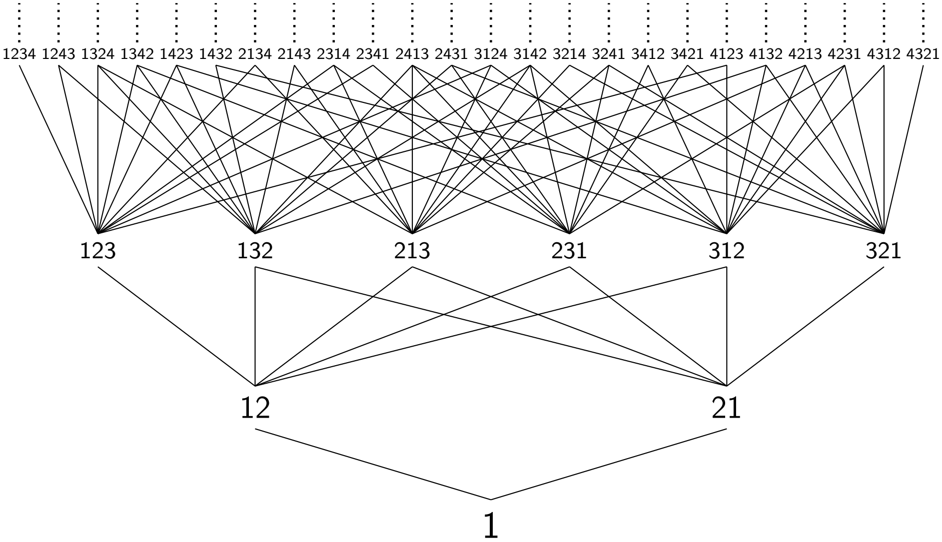
Theorem (Marcus and Tardos, 2004)

Every proper permutation class has a finite exponential growth rate. That is, for any proper class $\mathcal{C}$, there exists a real number s such that

$$\limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|} = s.$$

This number s is the *growth rate* of the class.

Permutation Classes - Growth Rates



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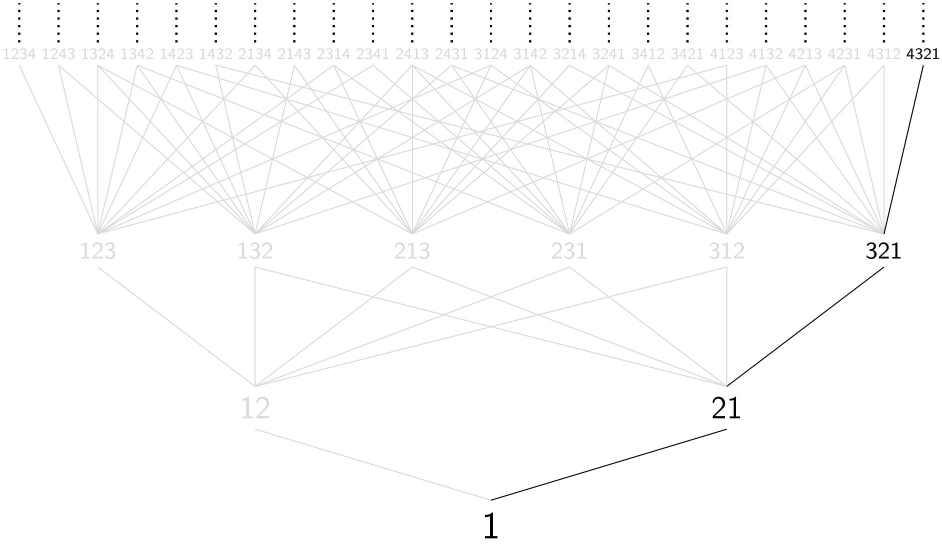

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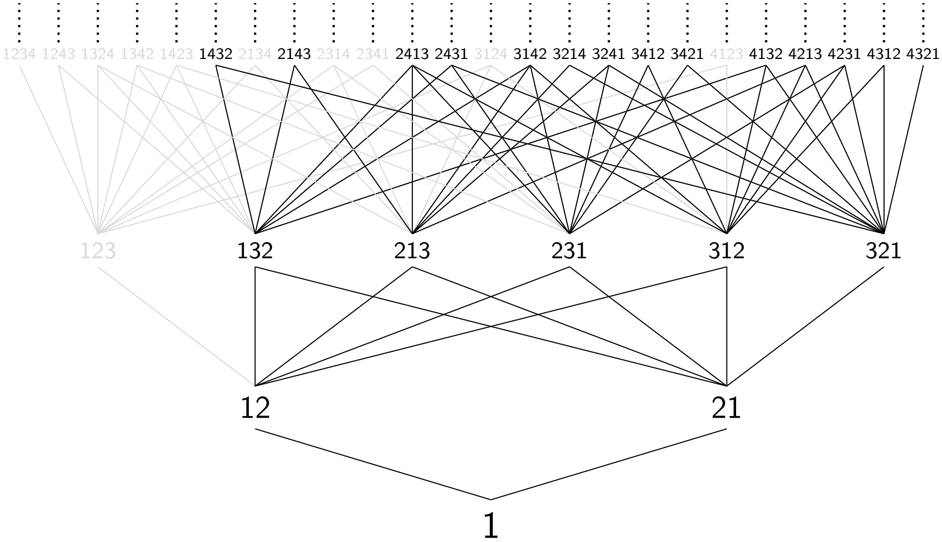

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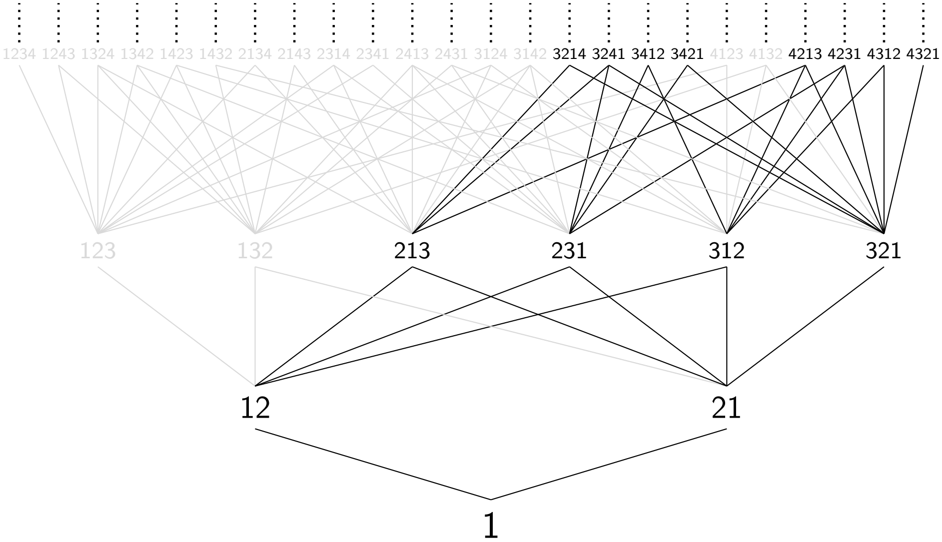

The Class Av(132)

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Let c_n be the number of permutations of length n which *avoid* the pattern 132, and $C(x) = \sum_{n \geq 0} c_n x^n$.

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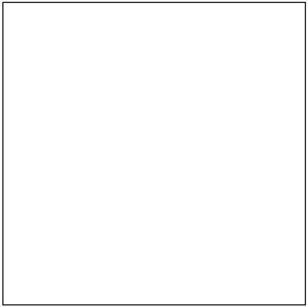

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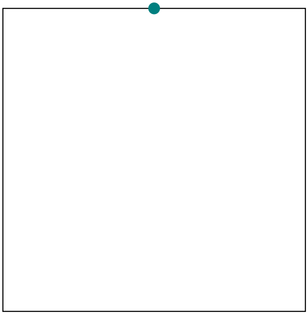

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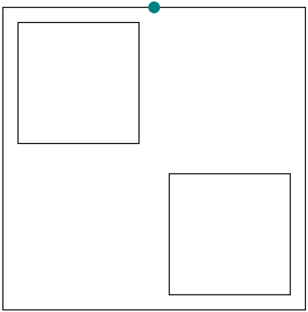

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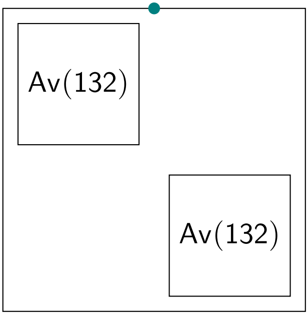

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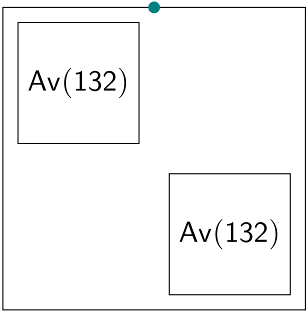

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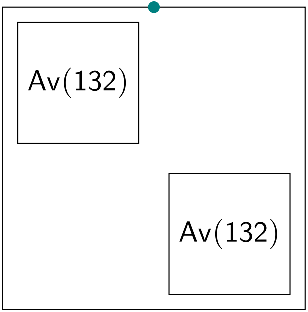

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$$0 = xC(x)^2 - C(x) + 1$$

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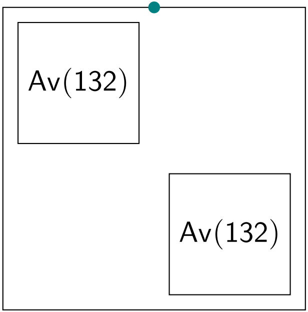

$$C(x) = xC(x)^2 + 1$$

$$0 = xC(x)^2 - C(x) + 1$$

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

The Class Av(132)

Definition

Let c_n be the number of permutations of length n which *avoid* the pattern 132, and $C(x) = \sum_{n\geq 0} c_n x^n$.

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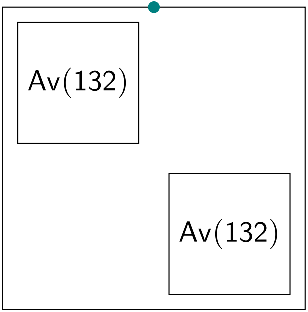

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$$C(x) = \tfrac{1-\sqrt{1-4x}}{2x}$$

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

The Class Av(123)

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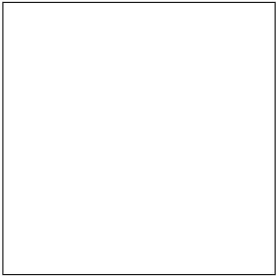

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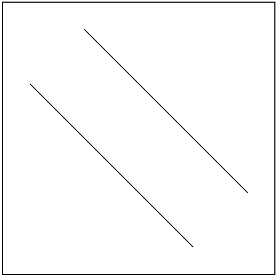

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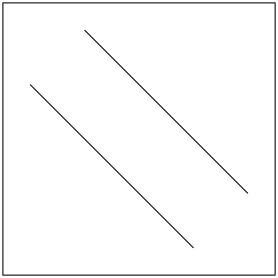

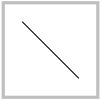

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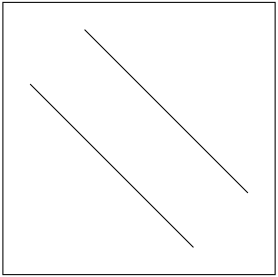

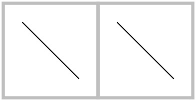

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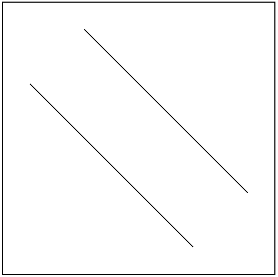

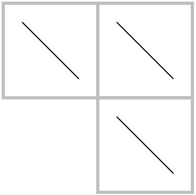

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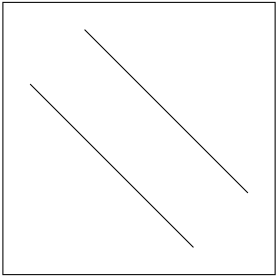

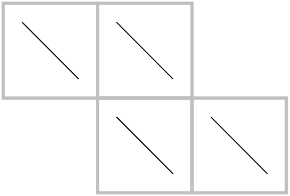

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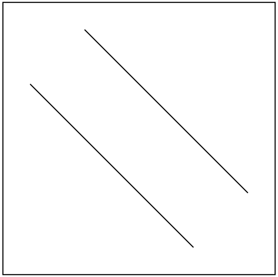

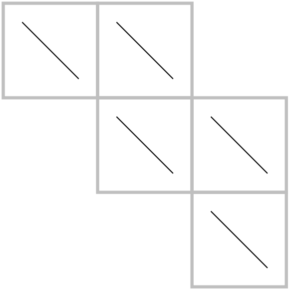

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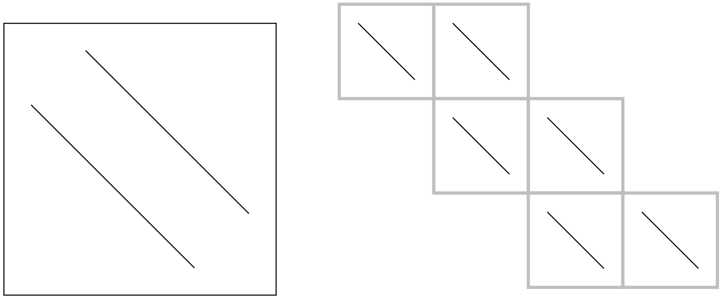

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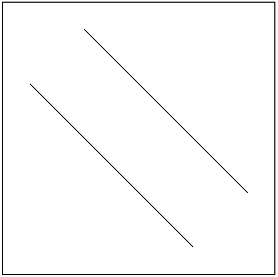
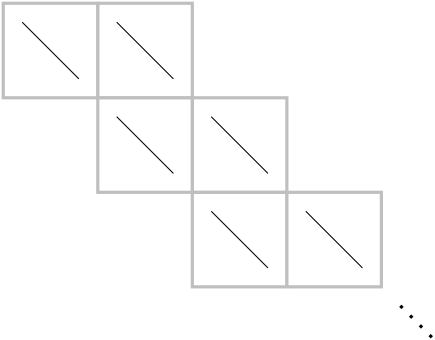
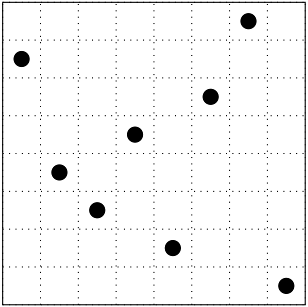




Av(132) and Av(123)



Av(132)

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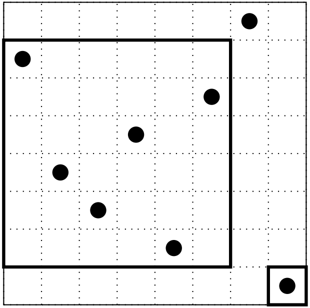

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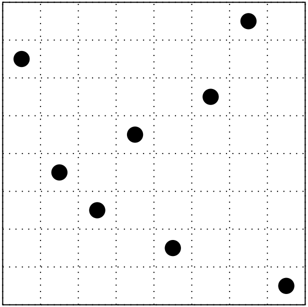

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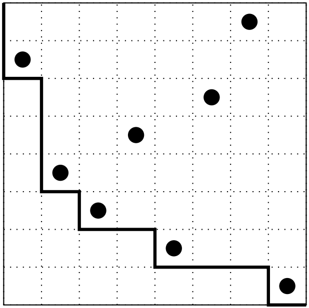

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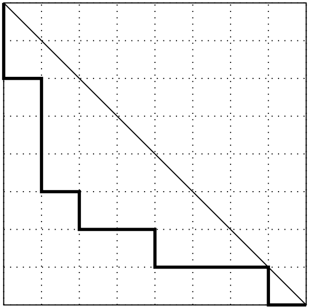

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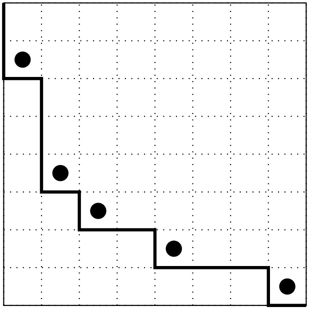

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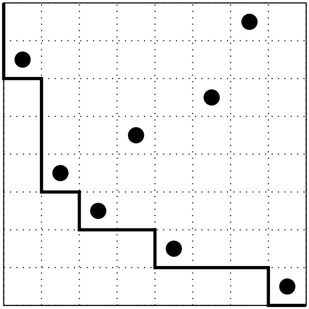

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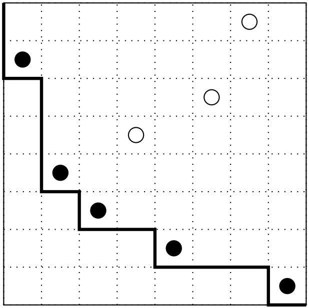

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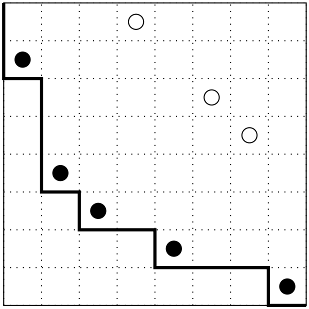
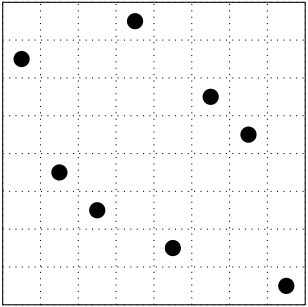


$\mathsf{Av}(132) \mapsto \mathsf{Av}(123)$

Av(132) and Av(123)



Av(123)

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$$|\operatorname{Av}_n(1324) = ???|$$

Pattern Occurrences

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Patterns

Say that one permutation π contains another permutation σ *as a pattern* (denoted $\sigma \prec \pi$) if the plot of π contains a subset which is equivalent to the plot of σ. The number of occurrences of σ in π (denoted $\nu_\sigma(\pi)$) is the number of such subsets.

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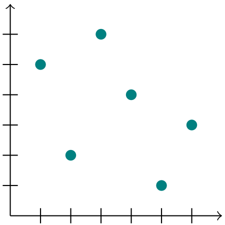

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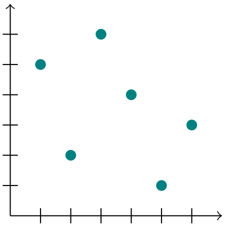

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$$\nu_{132}(526413) = 3$$

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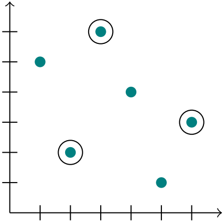

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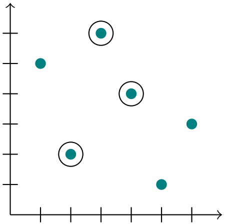

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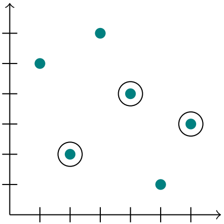

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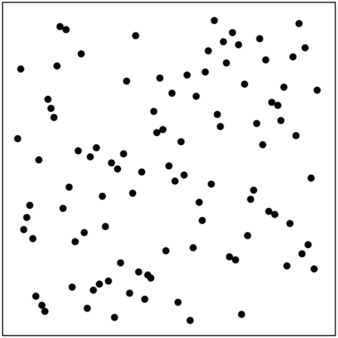
Random Data



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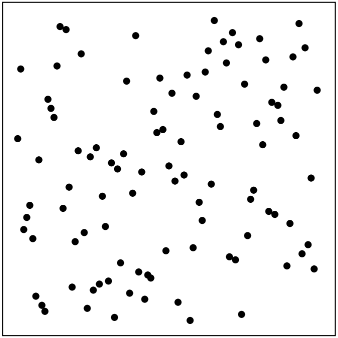

ν_{12}	ν_{21}	Avg
2803	2147	2475

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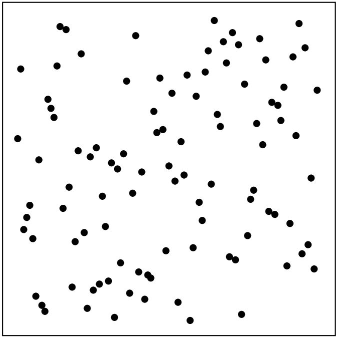

ν_{12}	ν_{21}	Avg
2803	2147	2475

ν_{123}	ν_{132}	ν_{213}	ν_{231}	ν_{312}	ν_{321}	Avg
35357	30063	31414	22321	23348	19197	26950

Patterns as Random Variables

Theorem (Bóna 2007)

For a (uniformly) randomly selected permutation of length n, the random variables ν_σ are asymptotically normal as n approaches infinity.

Theorem (Janson, Nakamura, Zeilberger 2013)

For a randomly selected permutation of length n and two patterns σ and ρ, the random variables ν_σ and ν_ρ are asymptotically jointly normally distributed as $n \to \infty$.

Motivation

Fact

In $\mathfrak{S}_n$, the number of occurrences of a specific pattern depends only on the length of the pattern. That is, for a pattern $\sigma \in \mathfrak{S}_k$, we have

$$\nu_\sigma(\mathfrak{S}_n) = \frac{n!}{k!}\binom{n}{k}.$$

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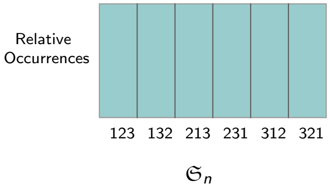

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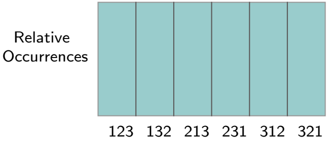
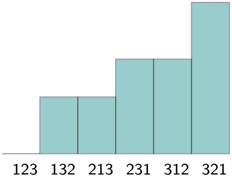


$\mathfrak{S}_n$



$\mathrm{Av}_n\,123$

Connections Between Classes

Av(123) and Av(132)

Previous Results

Theorem (Bóna 2010)

In $\mathrm{Av}_n\,132$, the pattern 123 is the least common, 321 is the most common, and $\nu_{213} = \nu_{231} = \nu_{312}$.

Data

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
4	10	0	11	11	11	13
5	68	0	81	81	81	109
6	392	0	500	500	500	748
7	2063	0	2794	2794	2794	4570

Data

Av 132

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Av 123

length	123	132	213	231	312	321
3	0	1	1	1	1	1
4	0	9	9	11	11	16

Data

Av 132

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Av 123

length	123	132	213	231	312	321
3	0	1	1	1	1	1
4	0	9	9	11	11	16
5	0	57	57	81	81	144
6	0	312	312	500	500	1016
7	0	1578	1578	2794	2794	6271

Patterns Within Av(123)

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Theorem (H 2012)

The total nuber of 231 (and 312) patterns is identical within the sets $\mathrm{Av}_n(123)$ and $\mathrm{Av}_n(132)$.

Patterns Within Av(123)

Theorem (H 2012)

The total nuber of 231 (and 312) patterns is identical within the sets $\mathsf{Av}_n(123)$ and $\mathsf{Av}_n(132)$.
Further, within $\mathsf{Av}_n(123)$,

$$\nu_{132} = \nu_{213} \sim \sqrt{\frac{n}{\pi}}4^n,$$
$$\nu_{231} = \nu_{312} \sim \frac{n}{2}4^n,$$
$$\text{and} \quad \nu_{321} \sim \frac{8}{3}\sqrt{\frac{n^3}{\pi}}4^n.$$

Sketch of Proof: Patterns in Av(123)

Sketch of Proof: Patterns in Av(123)

ν_{132} ν_{213} ν_{231} ν_{312} ν_{321}

Sketch of Proof: Patterns in Av(123)

$$\nu_{132} + \nu_{213} + \nu_{231} + \nu_{312} + \nu_{321} = \binom{n}{3} c_n$$

(Both sides count the number of length three patterns)

Sketch of Proof: Patterns in Av(123)

$$2\nu_{132} + 2\nu_{213} + \nu_{231} + \nu_{312} = (n-2)\nu_{12}$$

(Count triples containing a 12 pattern . . .)

Sketch of Proof: Patterns in Av(123)

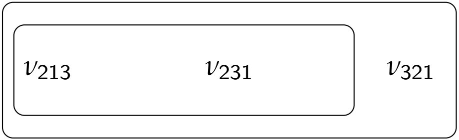
ν_{132} ν_{213} ν_{231} ν_{312} ν_{321}

Sketch of Proof: Patterns in Av(123)

$\nu_{132} = \nu_{213}$ $\nu_{231} = \nu_{312}$ ν_{321}

(Since Av(123) is closed under inversion)

Sketch of Proof: Patterns in Av(123)



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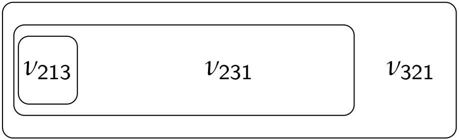

Sketch of Proof: Counting 213 Patterns

Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.

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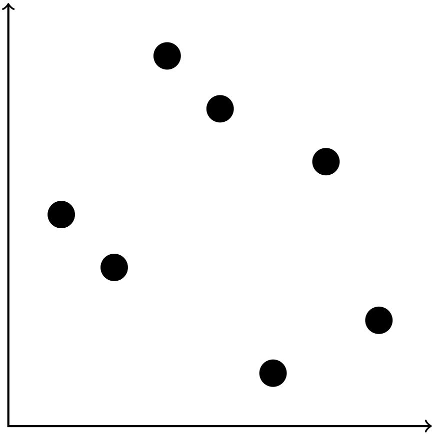

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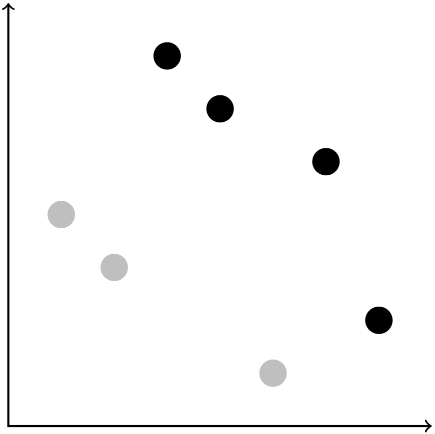

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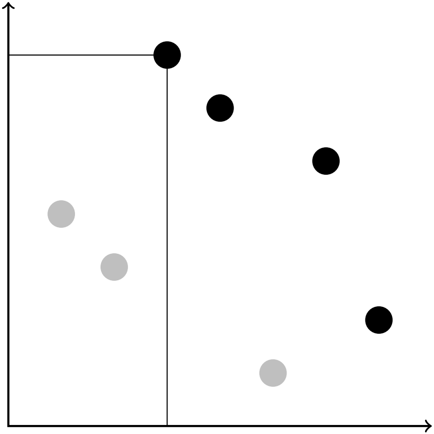
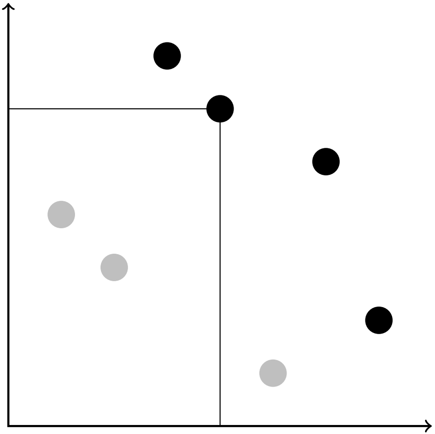


$$\nu_{213}(p) = \binom{2}{2}$$

Sketch of Proof: Counting 213 Patterns

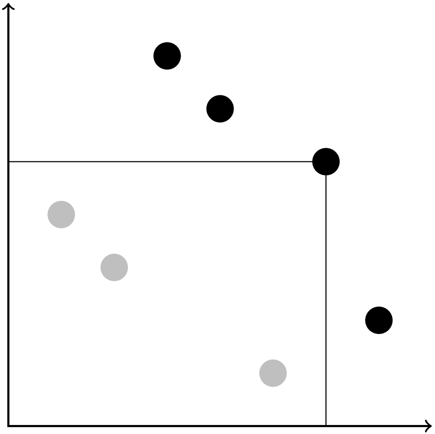
Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.



$$\nu_{213}(p) = \binom{2}{2} + \binom{2}{2}$$

Sketch of Proof: Counting 213 Patterns

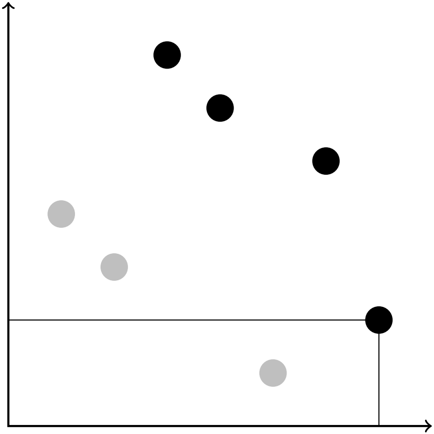
Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.



$$\nu_{213}(p) = \binom{2}{2} + \binom{2}{2} + \binom{3}{2}$$

Sketch of Proof: Counting 213 Patterns

Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.



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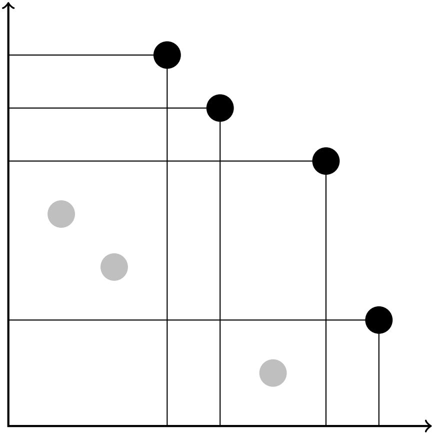

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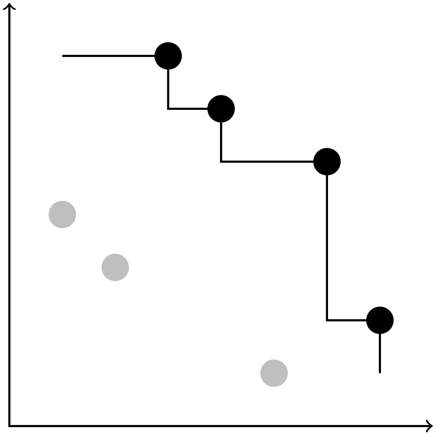

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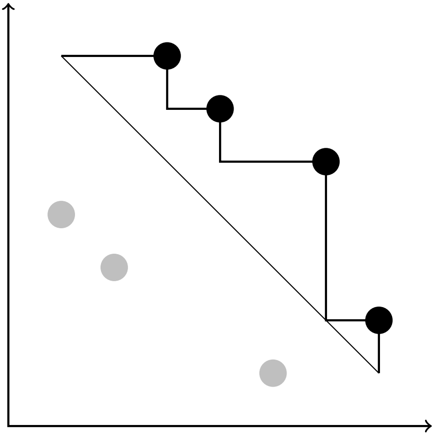

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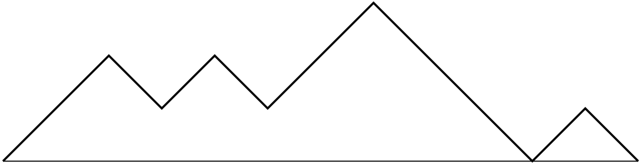

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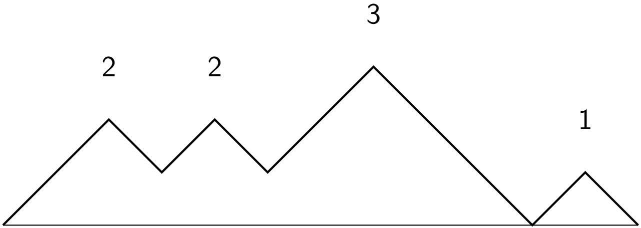

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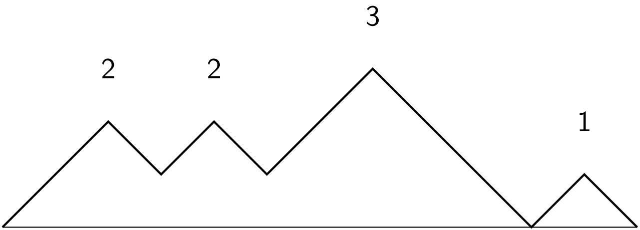
Sketch of Proof: Counting 213 Patterns

Let $h_{n,k}$ denote the total number of peaks at height k in all Dyck paths of semilength n. Let $H(x,u) = \sum_{n,k\geq 0} h_{n,k} x^n u^k$.



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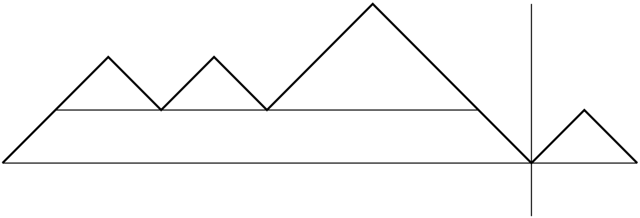

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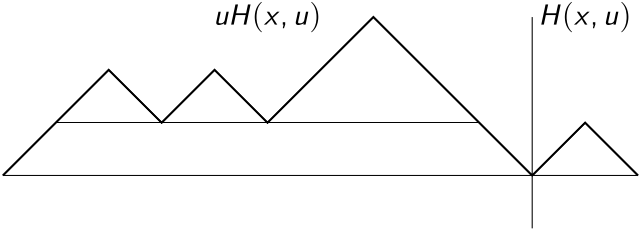

$$H(x,u) = ux(H(x,u)+1)C(x) + xC(x)H(x,u)$$

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□

Results

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$$\nu_{231}(\mathsf{Av}_n\, 123) = \nu_{231}(\mathsf{Av}_n\, 132)$$

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$$\nu_{213} = \frac{n+2}{4}\binom{2n}{n} - 3 \cdot 2^{2n-3}$$

$$\nu_{231} = (2n-1)\binom{2n-3}{n-2} - (2n+1)\binom{2n-1}{n-1} + (n+4) \cdot 2^{2n-3}$$

$$\begin{aligned}\nu_{321} = {} & \frac{1}{6}\binom{2n+5}{n+1}\binom{n+4}{2} - \frac{5}{3}\binom{2n+3}{n}\binom{n+3}{2} \\ & + \frac{17}{3}\binom{2n+1}{n-1}\binom{n+2}{2} - 6\binom{2n-1}{n-2}\binom{n+1}{2} - (n+1) \cdot 4^{n-1}.\end{aligned}$$

Connections Within Classes

Av(132) and the Separables

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Corollary

The equipopularity classes within Av(132) are in bijection with the set of integer partitions.

Separable Permutations

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Theorem (Albert, Pantone, H 2014)

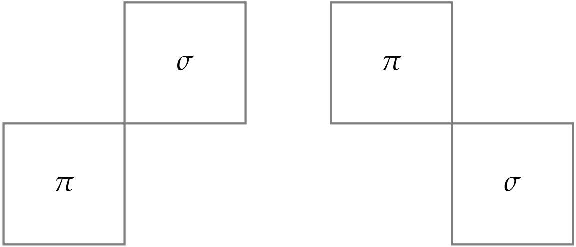
Two patterns are equipopular in the separables if and only if they *have the same structure*.

Separable Permutations

Separable Permutations

Definition

Given two permutations π and σ, their *direct sum* $(\pi \oplus \sigma)$ and *skew sum* $(\pi \ominus \sigma)$ are defined as follows:



$\pi \oplus \sigma$ $\qquad$ $\pi \ominus \sigma$

Separable Permutations

Alternate Definition

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Example

The permutation $\pi = 215643798$ is separable, since

$$\pi = \Big(1 \ominus 1\Big) \oplus \Big((1 \oplus 1) \ominus 1 \ominus 1\Big) \oplus 1 \oplus \Big(1 \ominus 1\Big).$$

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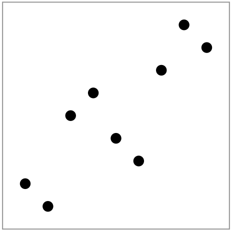

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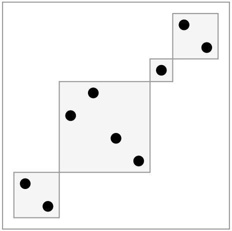

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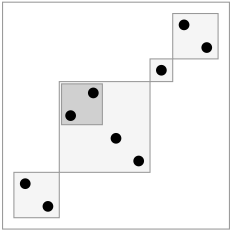

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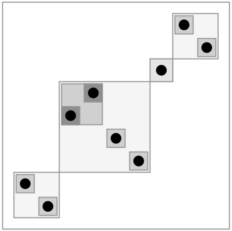

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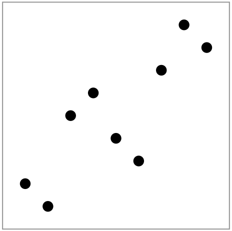

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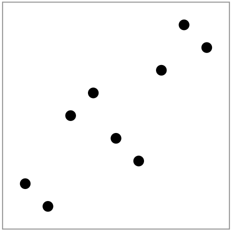

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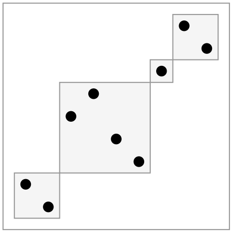

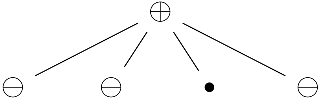

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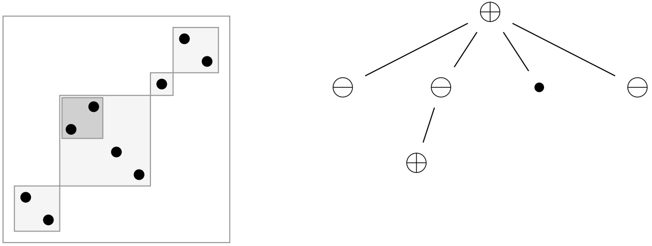

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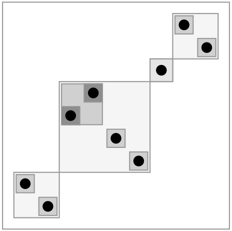

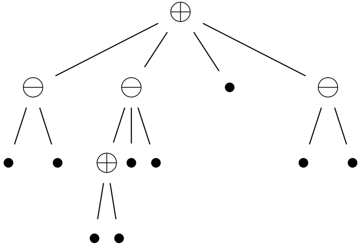

Tree Containment

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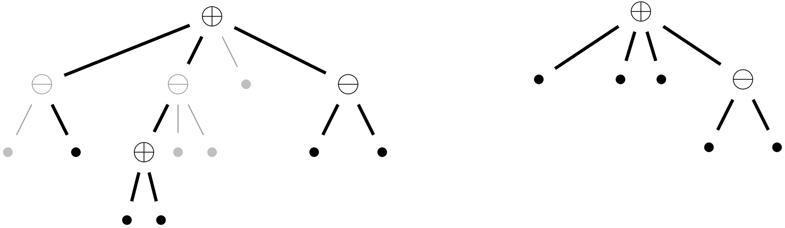

Equipopularity

Question

If two patterns are equipopular, how are their trees related?

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Question

What tree transformations preserve equipopularity?

Strategy

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Part 1

Find the operations on trees which preserve popularity.

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Find the operations on trees which preserve popularity.

Part 2
Show that equipopularity implies that their trees are related by one of these operations.

Preserving Popularity

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Symmetries

Permutation	Tree

Preserving Popularity

Symmetries

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Complementation	Flip signs

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Fact

If two permutations (trees) are related by any of the above symmetries, then they are equipopular.

Preserving Popularity - Shuffling

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Theorem (Albert, Pantone, H 2014)

Rearranging the children of any node preserves equipopularity.

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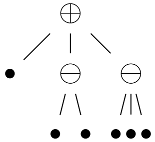

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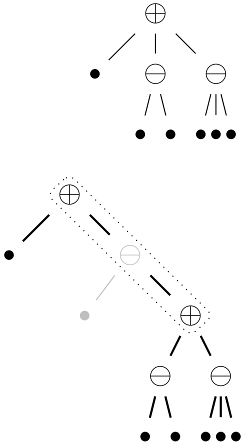

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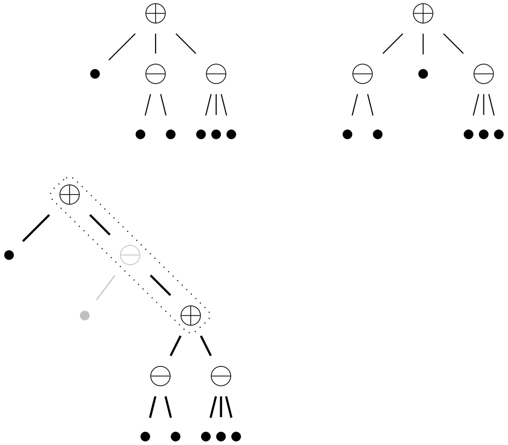

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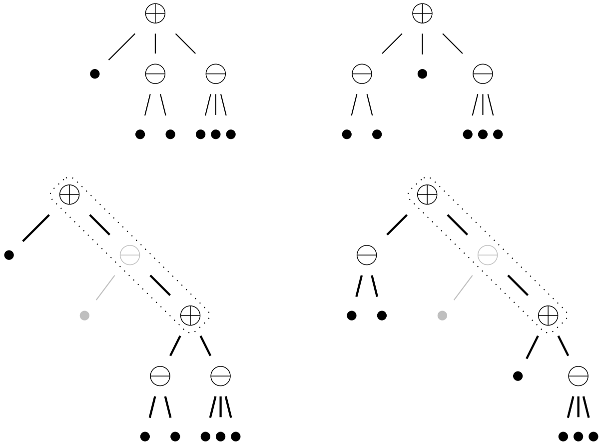
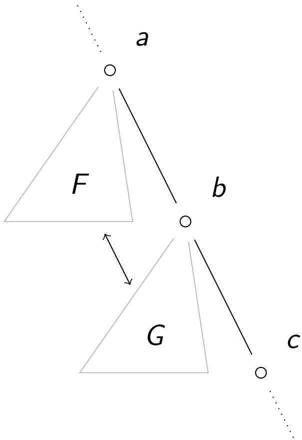


Preserving Popularity - Rotation



Preserving Popularity

Theorem (Albert, Pantone, H 2014)

The following operations preserve popularity:

- Reversal
- Complementation
- Inversion
- Shuffling
- Rotation

Canonical Representatives

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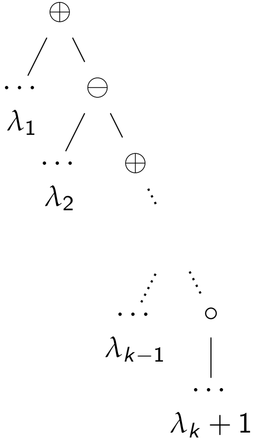

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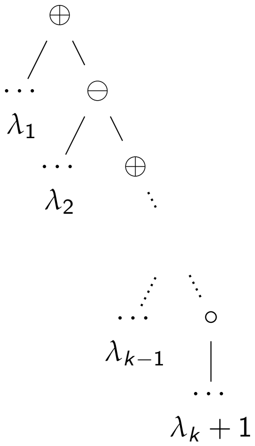

$$\lambda := \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

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If two patterns are equipopular, one can be transformed into the other by the above operations.

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Corollary

The set of equipopularity classes for patterns of length n are in bijection with the set of partitions of the integer $n - 1$.

Rough Sketch of Proof

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Given any arbitrary pattern, we can factor its popularity generating function into the popularity generating functions for monotone runs.

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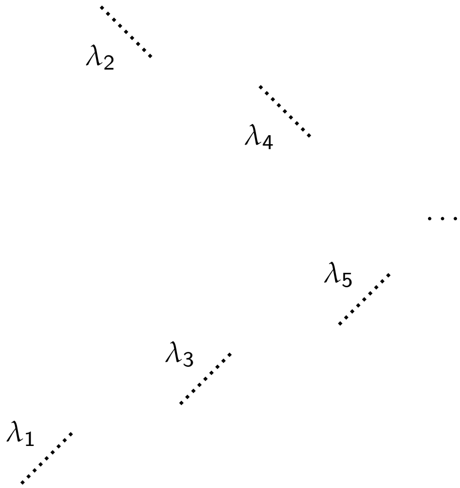

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- Notice (or let Sage tell you) that these are related to the *Gegenbauer polynomials*, a family of orthogonal polynomials.
- Use the orthogonality of these polynomials to uniquely factor any product.

Future Work

What else?

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Similar results within other classes?

What else?

Similar results within other classes? Similar connections between classes?

What else?

Similar results within other classes? Similar connections between classes? Beyond averages: equidistribution?

Thank you for your afternoon!