Patterns Within Random Permutations

Some Open (and Recently Closed) Questions

Cheyne Homberger

January 22, 2014

Permutations: Representation/Notation

Definition

An *n-permutation* is a bijection $p : [n] \to [n]$.
The set of all n-permutations is denoted by $\mathfrak{S}_n$.

Two/One-Line Notation

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 5 & 1 & 4 & 2 \end{array}$$

Plotting Permutations

Definition

If $\pi = \pi_1\pi_2 \cdots \pi_n$ is a permutation written in one line notation, then the *plot* of π is the set of points

$$\{(1, \pi_1), (2, \pi_2), \cdots (n, \pi_n)\} \subset \mathbb{R}^2$$

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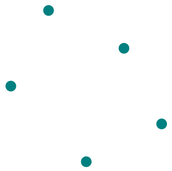

$\pi = 35142$

Dots on a Plane

Definition

Let A and B be two sets of n points in $\mathbb{R}^2$, each with the property that no two points lie on the same horizontal or vertical line.
Say that $A \sim B$ if A can be transformed into B by stretching, contracting, and translating the axes horizontally and vertically.

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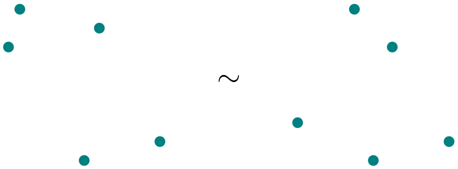

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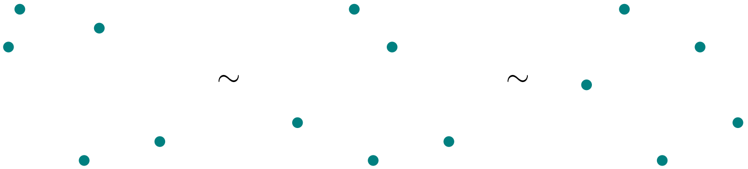

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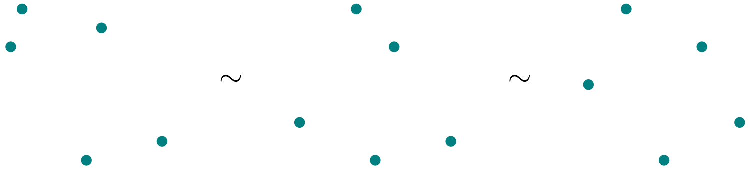

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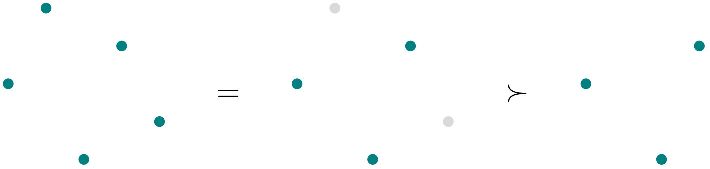

$$35142 \quad \succ \quad 213$$

The Pattern Poset

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12 21

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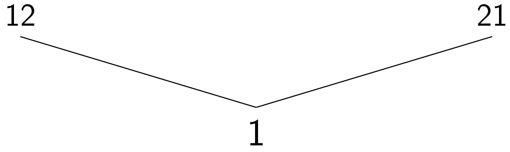
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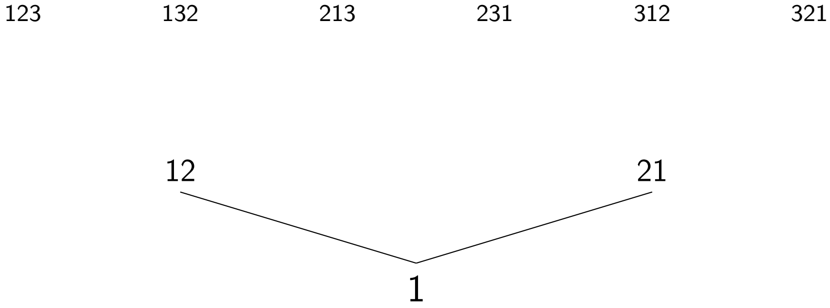

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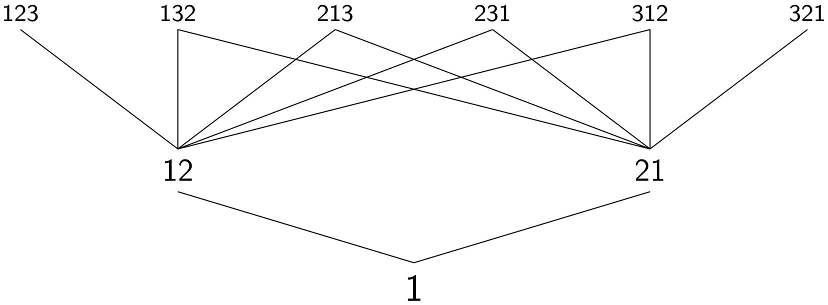

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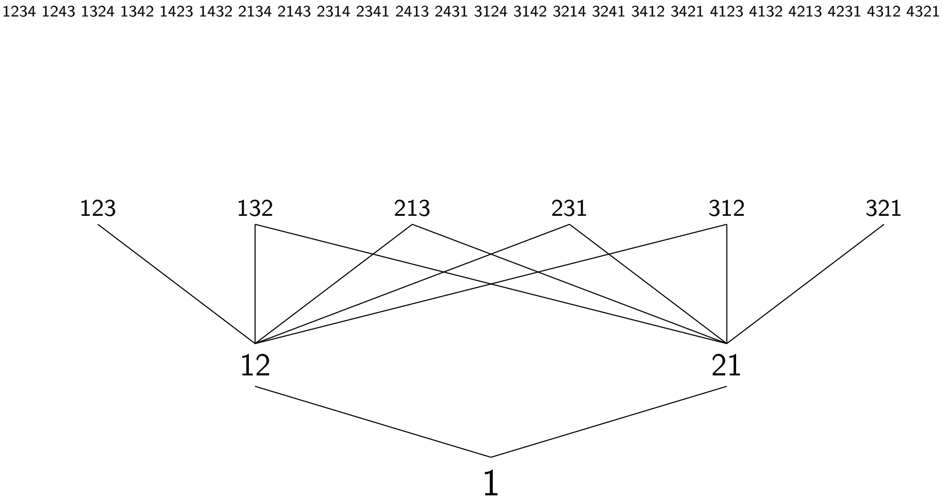

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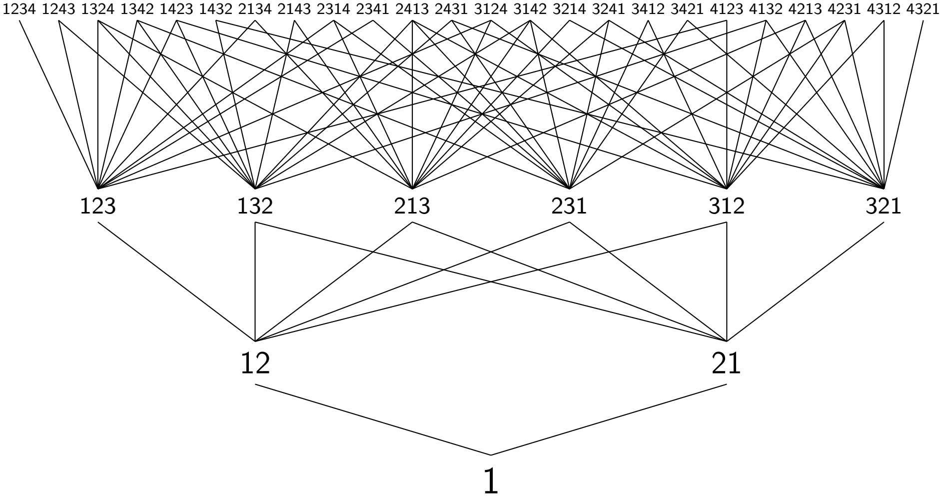

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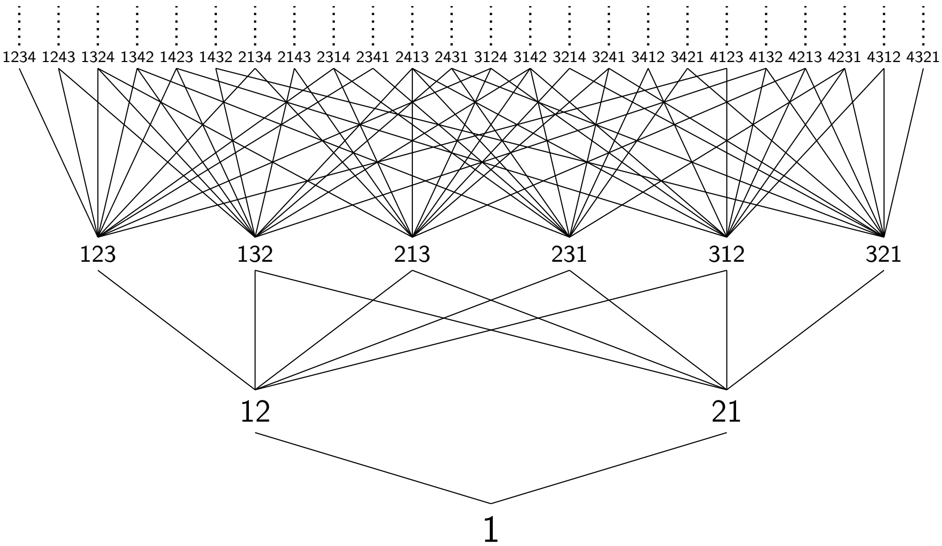

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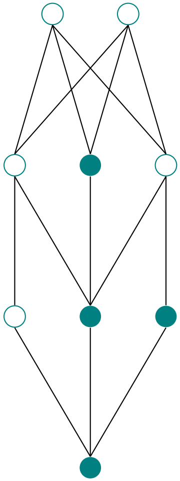
Let $\mathcal{P}$ be a poset, and $A \subset \mathcal{P}$. A is a *downset* if it is closed downwards (i.e., $x \in A$ and $y < x$ implies $y \in A$).
An *upset* is a subset which is closed upwards.

Fact

The complement of a downset is an upset.



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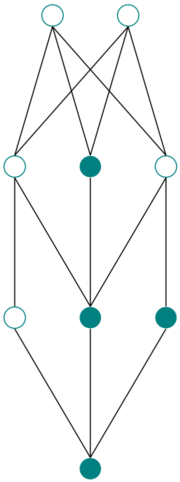
A downset in the permutation pattern poset is called a *permutation class*.



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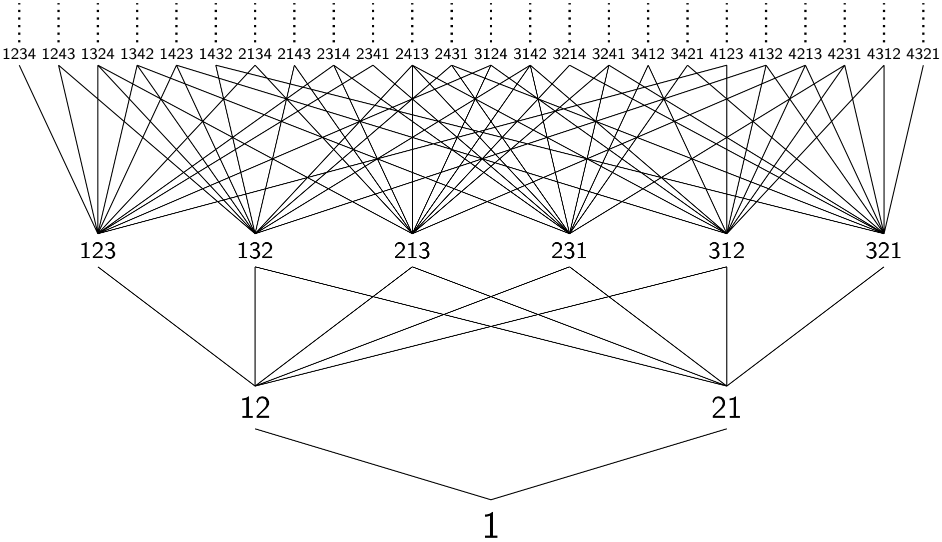

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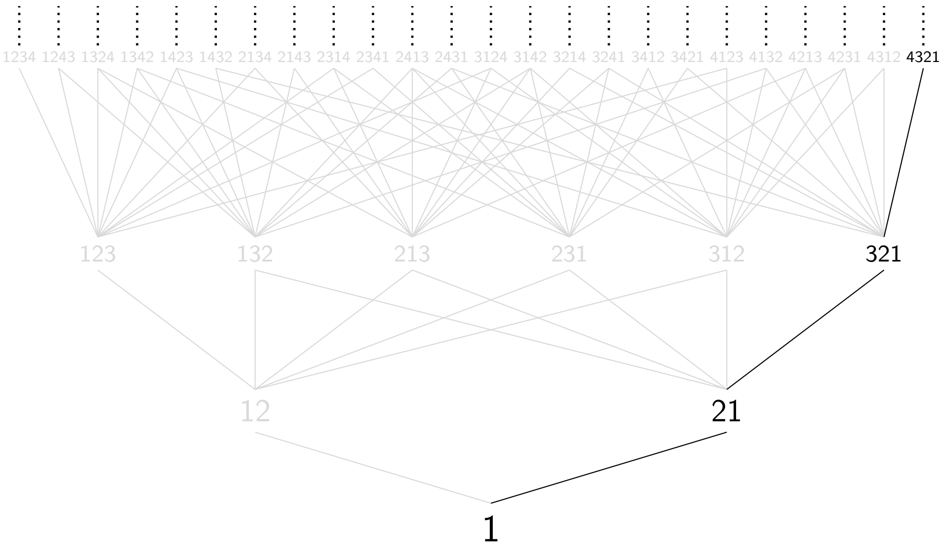

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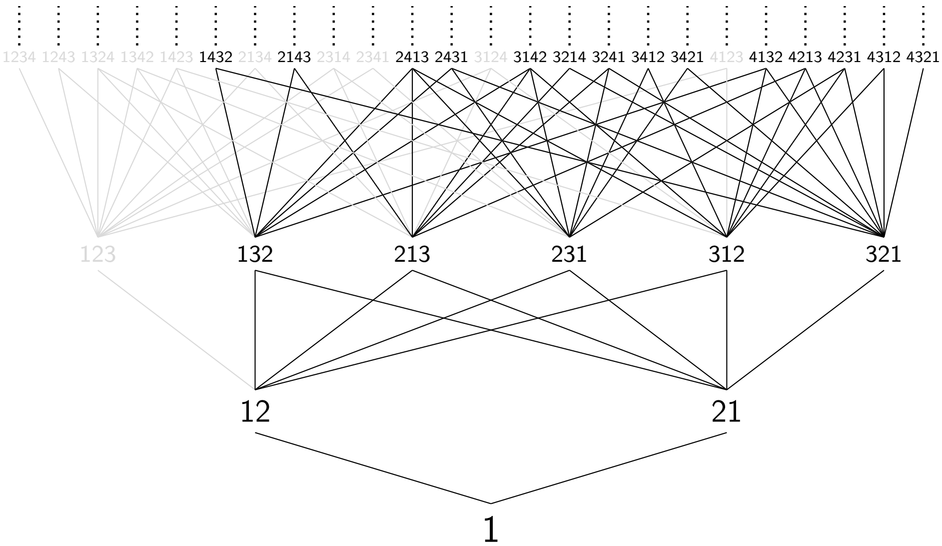

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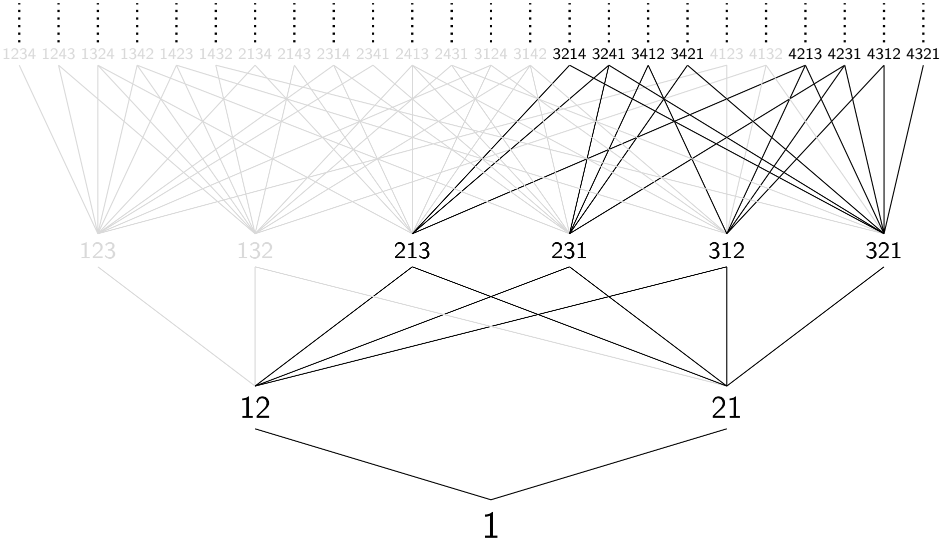

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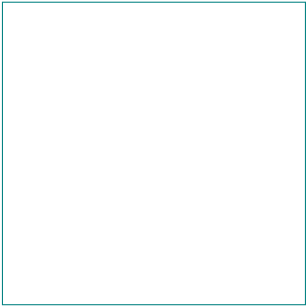

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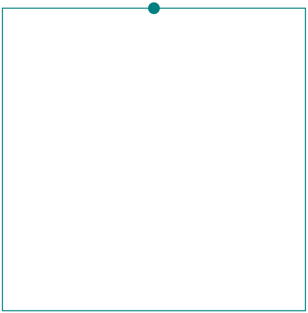

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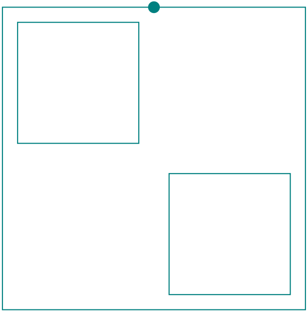

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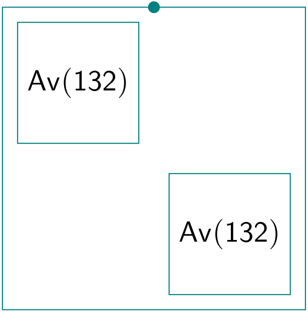

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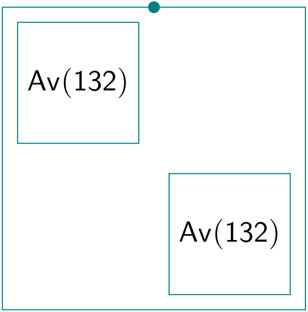

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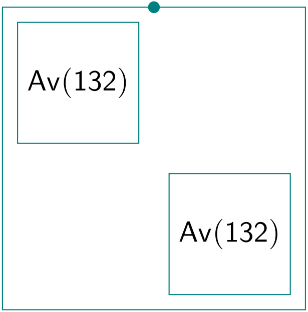

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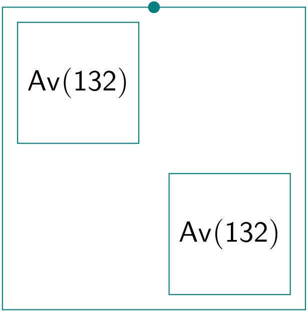

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$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

Lattice Paths

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Definition

A *NS Lattice path* of length $2n$ (or semilength n) is a sequence of vectors from the set $\{\langle 1,1\rangle, \langle 1,-1\rangle\}$ such that their sum is $\langle 2n, 0\rangle$.

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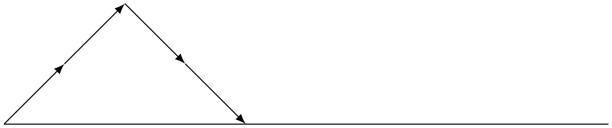

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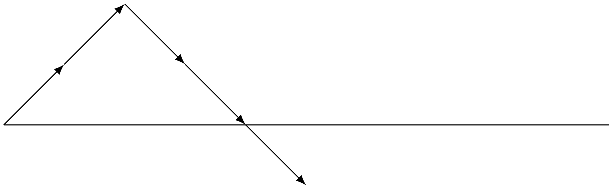

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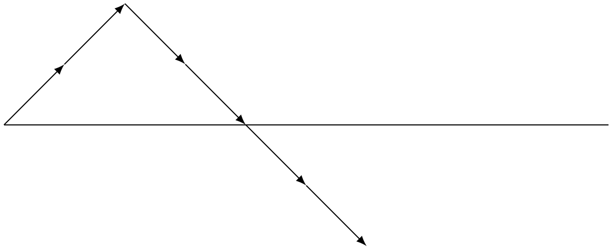

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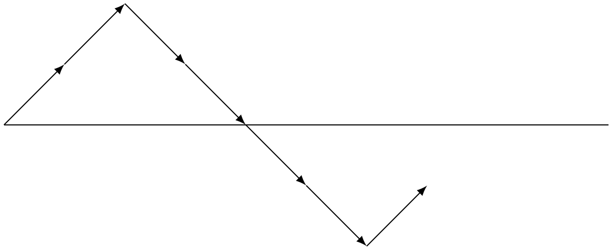

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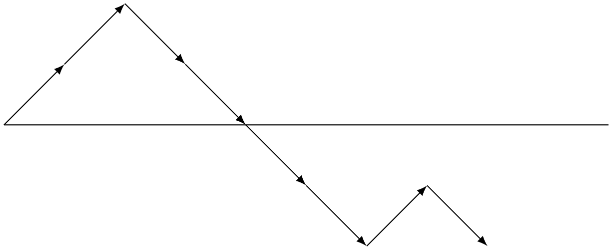

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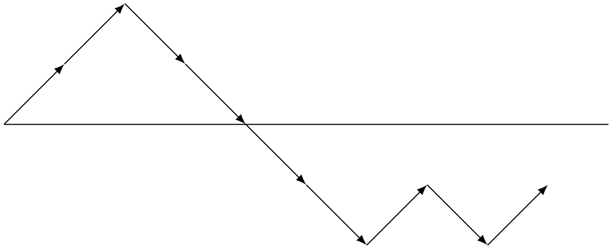

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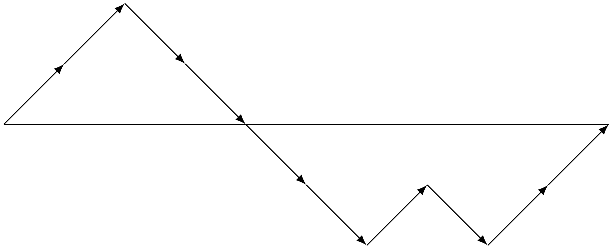

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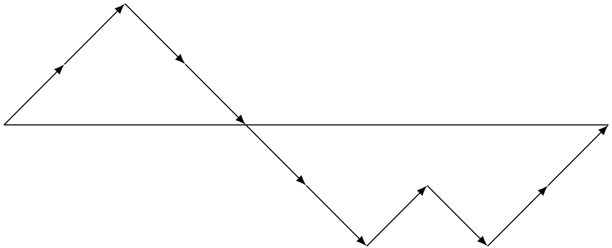

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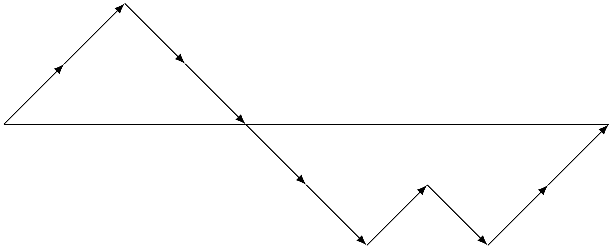

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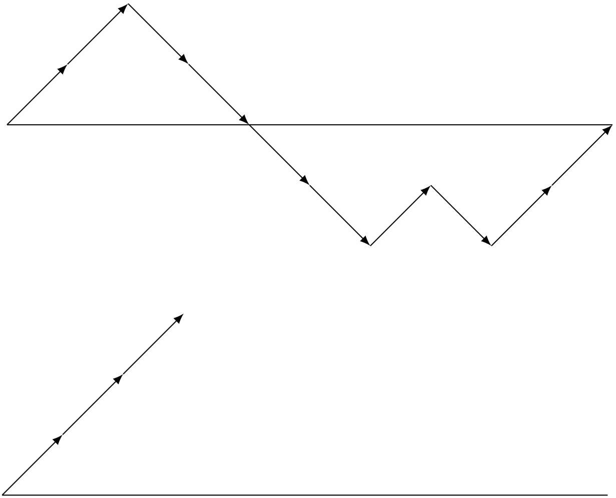

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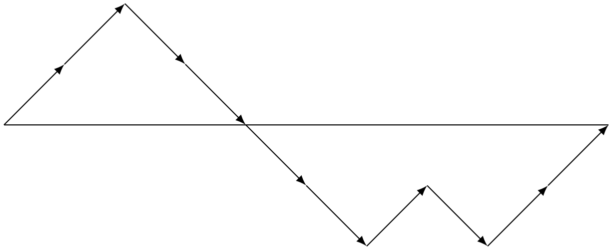

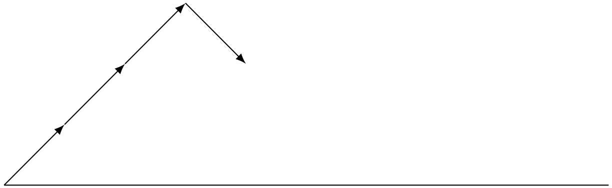

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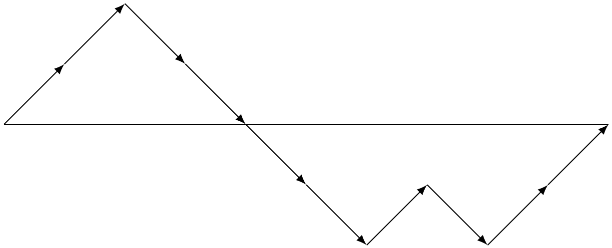

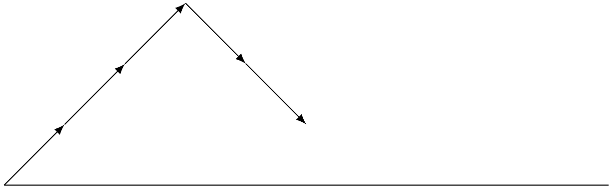

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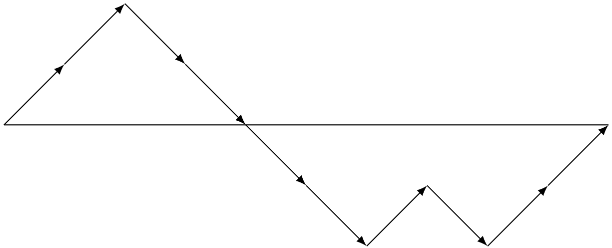

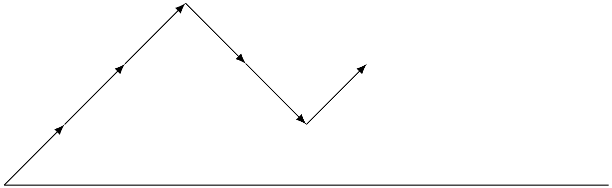

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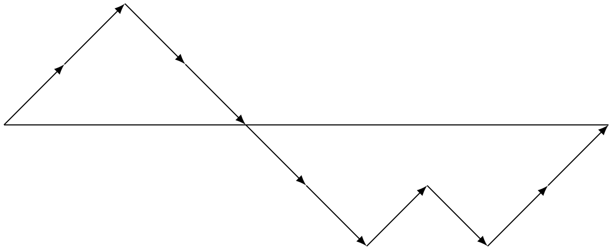

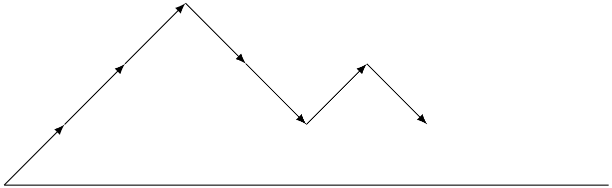

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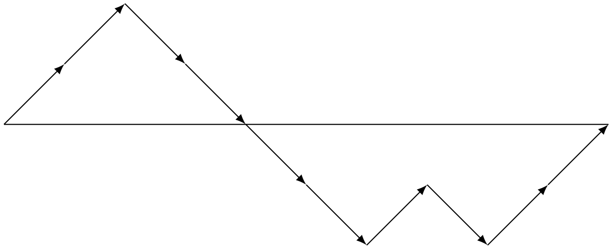

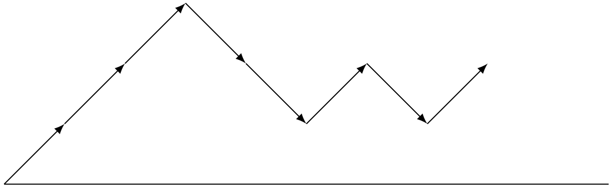

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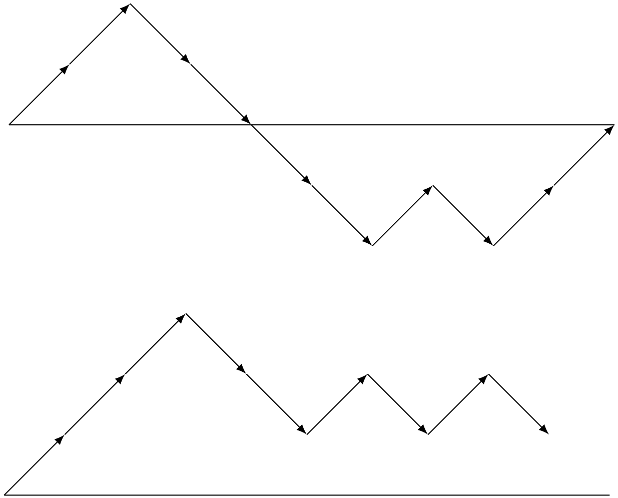

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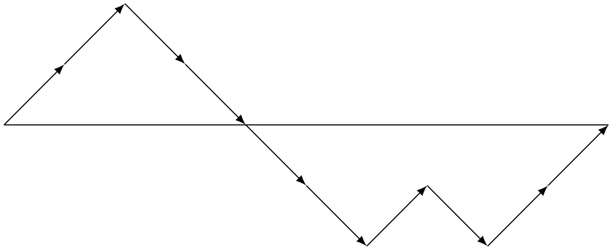

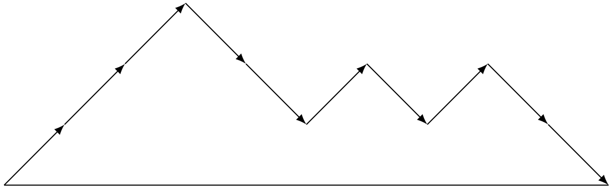

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Q2) How many NS lattice paths are there of semilength n which never pass below the line $y = 0$?

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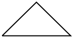

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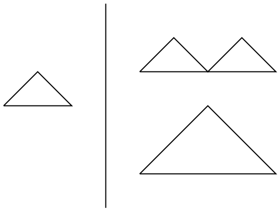

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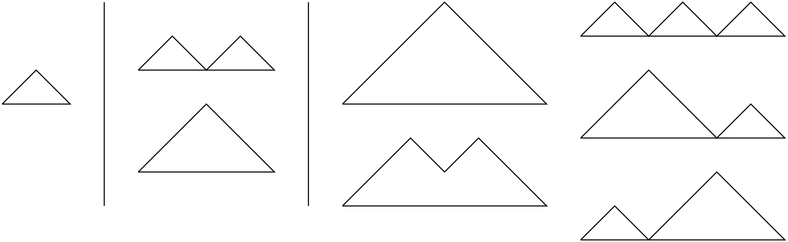

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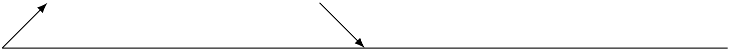

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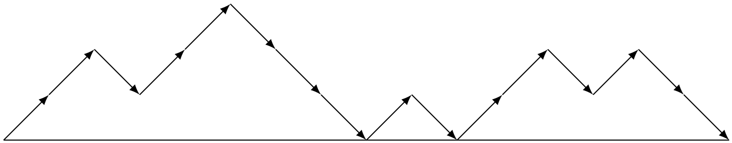

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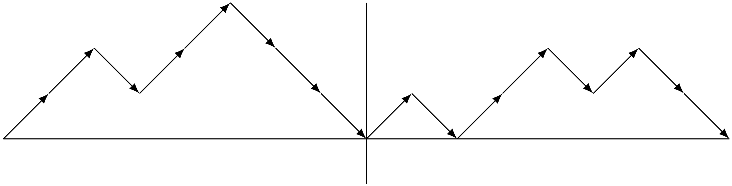

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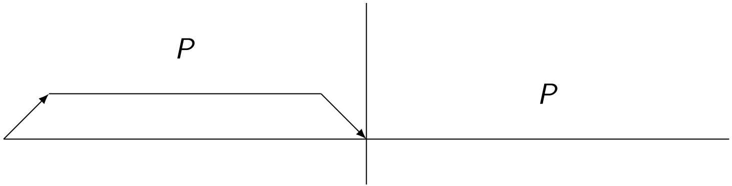
Let p_n be the number of NS paths of length $2n$ that *don't* cross below the x-axis, and let $P = \sum_{n \geq 0} p_n x^n$.



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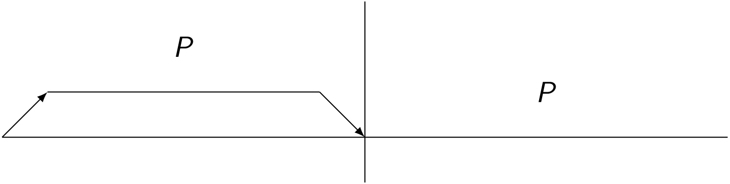

$P = xP^2 + 1$

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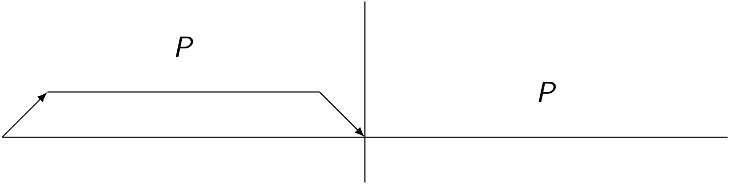

$P = xP^2 + 1 \qquad 0 = xP^2 - P + 1$

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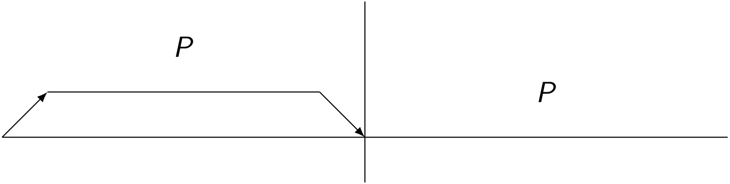

$P = xP^2 + 1$ $\qquad$ $0 = xP^2 - P + 1$ $\qquad$ $P = \frac{1 \pm \sqrt{1-4x}}{2x}$

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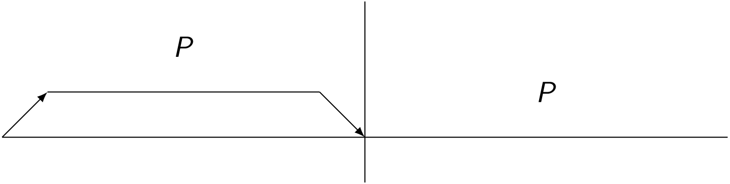

$$P = xP^2 + 1 \qquad 0 = xP^2 - P + 1 \qquad P = \tfrac{1 \pm \sqrt{1-4x}}{2x}$$

$$P = \frac{1 - \sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + \dots$$

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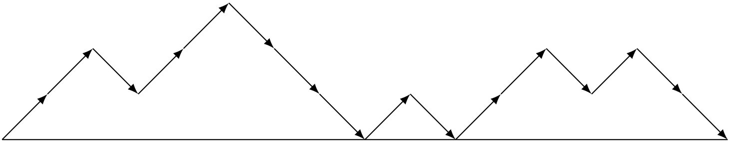

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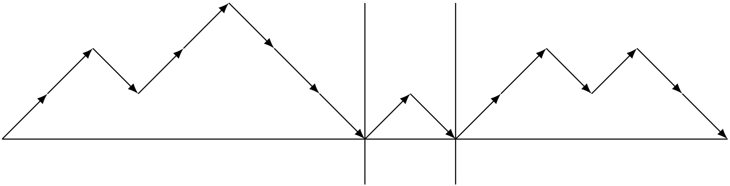

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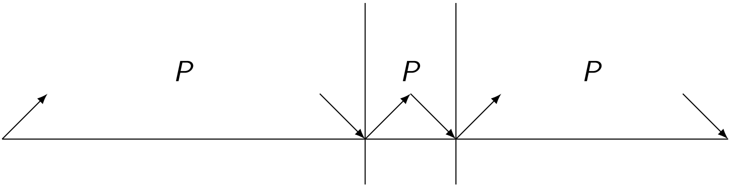

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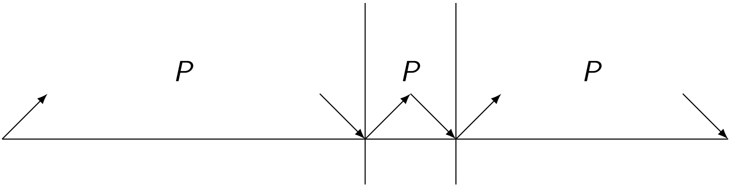

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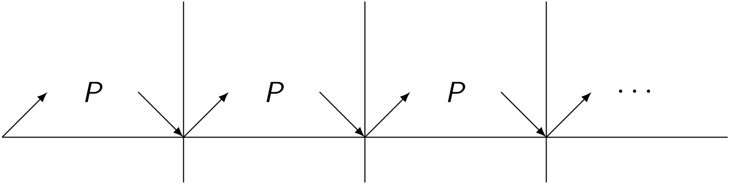

$$P = 1 + xP + x^2P^2 + x^3P^3 + x^4P^4 + x^5P^5 + \cdots$$

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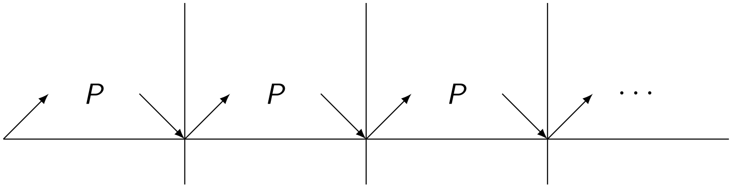

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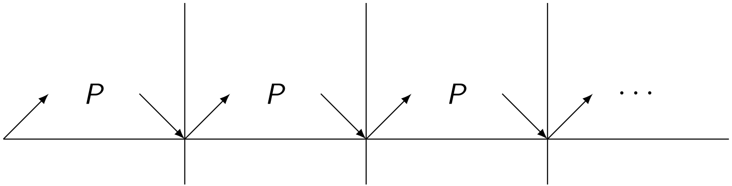

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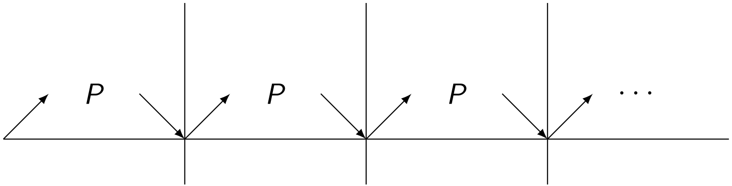

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Catalan Numbers

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Fact

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n.$$

Catalan Numbers

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The numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$ are called the Catalan numbers. The first few numbers in the sequence are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862 \cdots$$

Two Catalan Recurrences

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$$C(x) = xC(x)^2 + 1 \qquad \text{and} \qquad C(x) = 1 + xC(x) + x^2C(x) + \cdots$$

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$$c_0 = 1, \quad c_i = 0 \text{ for } i < 0$$

$$c_n = \sum_{i+j=n-1} c_i c_j$$

$$c_n = c_{n-1} + \sum_{i+j=n-2} c_i c_j + \sum_{i+j+k=n-3} c_i c_j c_k + \cdots$$

The Class Av(123)

Question

What does a 123-avoiding permutation look like?

The Class Av(123)

$$\pi = 4\ 7\ 6\ 2\ 5\ 1\ 3$$

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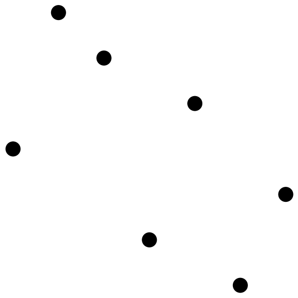

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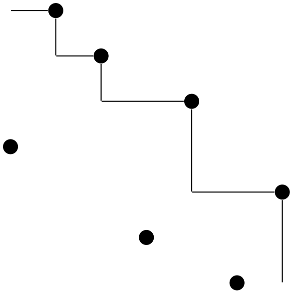

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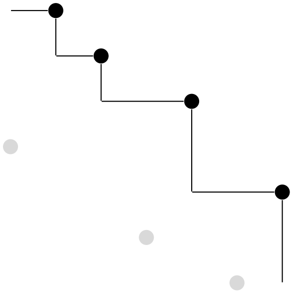

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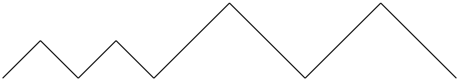

Counting Patterns

Introduction

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Example

The permutation 3 5 1 2 4 contains the pattern 3 1 2 three times.

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Example

The set $\{2341, 1234, 4321\}$ contains the pattern 123 exactly 5 times.

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Notation

Let $f_\sigma(S)$ denote the number of occurrences of σ within the set S.

An Example

Question

How many times does the pattern 1324 occur within the set of all n-permutations? That is, what is

$$f_{1324}(\mathfrak{S}_n)?$$

An Example

Answer

Let X be a random variable denoting the number of 1324 patterns in a random n-permutation. Then $\mathbb{E}[X] = f_{1324}(\mathfrak{S}_n)/n!$.

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Let X be a random variable denoting the number of 1324 patterns in a random n-permutation. Then $\mathbb{E}[X] = f_{1324}(\mathfrak{S}_n)/n!$.

Let $X_{i,j,k,l}$ be the random variable equal to 0 or 1, which indicates whether the ith, jth, kth, and lth entries form a 1324 pattern. Then

$$X = \sum_{1 \le i < j < k < l \le n} X_{i,j,k,l}$$

An Example

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Then

$$X = \sum_{1 \le i < j < k < l \le n} X_{i,j,k,l}$$

$$\mathbb{E}[X] = \sum_{1 \le i < j < k < l \le n} \mathbb{E}[X_{i,j,k,l}]$$

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Then

$$X = \sum_{1 \le i < j < k < l \le n} X_{i,j,k,l}$$

$$\begin{aligned} \mathbb{E}[X] &= \sum_{1 \le i < j < k < l \le n} \mathbb{E}[X_{i,j,k,l}] \\ &= \sum_{1 \le i < j < k < l \le n} \frac{1}{4!} \end{aligned}$$

An Example

Answer

Let X be a random variable denoting the number of 1324 patterns in a random n-permutation. Then $\mathbb{E}[X] = f_{1324}(\mathfrak{S}_n)/n!$.

Let $X_{i,j,k,l}$ be the random variable equal to 0 or 1, which indicates whether the ith, jth, kth, and lth entries form a 1324 pattern. Then

$$X = \sum_{1 \le i < j < k < l \le n} X_{i,j,k,l}$$

$$\mathbb{E}[X] = \sum_{1 \le i < j < k < l \le n} \mathbb{E}[X_{i,j,k,l}]$$

$$= \sum_{1 \le i < j < k < l \le n} \frac{1}{4!} = \binom{n}{4} \frac{1}{4!}.$$

Therefore

$$f_{1324}(\mathfrak{S}_n) = \binom{n}{4} \frac{n!}{4!}.$$

Motivation

Fact

In $\mathfrak{S}_n$, the number of occurrences of a specific pattern depends only on the length of the pattern.

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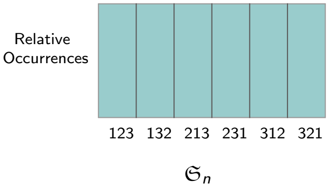

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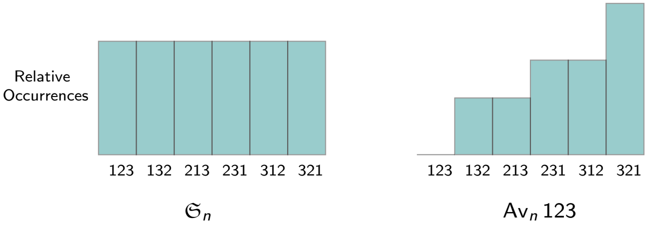

Data

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
4	10	0	11	11	11	13

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
4	10	0	11	11	11	13
5	68	0	81	81	81	109
6	392	0	500	500	500	748
7	2063	0	2794	2794	2794	4570

Previous Results

Theorem (Bóna)

In $\mathrm{Av}_n\,132$, the pattern 123 is the least common, 321 is the most common, and $f_{213} = f_{231} = f_{312}$.

Data

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
4	10	0	11	11	11	13
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6	392	0	500	500	500	748
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Av 123

length	123	132	213	231	312	321
3	0	1	1	1	1	1

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
4	10	0	11	11	11	13
5	68	0	81	81	81	109
6	392	0	500	500	500	748
7	2063	0	2794	2794	2794	4570

Av 123

length	123	132	213	231	312	321
3	0	1	1	1	1	1
4	0	9	9	11	11	16

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
4	10	0	11	11	11	13
5	68	0	81	81	81	109
6	392	0	500	500	500	748
7	2063	0	2794	2794	2794	4570

Av 123

length	123	132	213	231	312	321
3	0	1	1	1	1	1
4	0	9	9	11	11	16
5	0	57	57	81	81	144
6	0	312	312	500	500	1016
7	0	1578	1578	2794	2794	6271

Patterns Within Av 123

Patterns of Length 2

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Theorem (Cheng, Eu, Fu)

$$f_{12}(\mathrm{Av}_n\, 123) = 4^{n-1} - \binom{2n-1}{n}.$$

Patterns of Length 2

Theorem (Cheng, Eu, Fu)

$$f_{12}(\mathrm{Av}_n\, 123) = 4^{n-1} - \binom{2n-1}{n}.$$

Fact

$$(f_{12} + f_{21})(\mathrm{Av}_n(123)) = \binom{n}{2} c_n.$$

Patterns of Length 3

Patterns of Length 3

Fact

$$2f_{132} + 2f_{231} + f_{321} = \binom{n}{3} c_n.$$

Patterns of Length 3

Fact

$$2f_{132} + 2f_{231} + f_{321} = \binom{n}{3} c_n.$$

Proof.

Rewrite the left hand side as

$$f_{132} + f_{213} + f_{231} + f_{312} + f_{321}$$

□

Patterns of Length 3

Proposition

$$(4f_{132} + 2f_{231})(\mathsf{Av}_n(123)) = (n-2)f_{12}(\mathsf{Av}_n(123)).$$

Patterns of Length 3

Proposition

$$(4f_{132} + 2f_{231})(\mathsf{Av}_n(123)) = (n-2)f_{12}(\mathsf{Av}_n(123)).$$

Proof.

Rewrite as

$$(n-2)f_{12} - f_{132} - f_{213} = f_{231} + f_{312} + f_{132} + f_{213}.$$

Both sides count the number of length three patterns with at least one non-inversion. □

Indecomposable Permutations

Indecomposable Permutations

Definition

We say that a permutation $p = p_1 p_2 \dots p_n$ is *decomposable* if there exists an integer k so that each of the entries $p_1, \dots p_k$ is greater than each of the entries $p_{k+1}, \dots p_n$. Otherwise, we say that p is *indecomposable*

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Example

The permutation 356412 is decomposable, as the entries 3564 are larger than the entries 12

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Example

The permutation 356412 is decomposable, as the entries 3564 are larger than the entries 12

Definition

Denote by $\mathrm{Av}_n^*(123)$ the set of indecomposable n-permutations which avoid 123.

Indecomposable Permutations

Fact

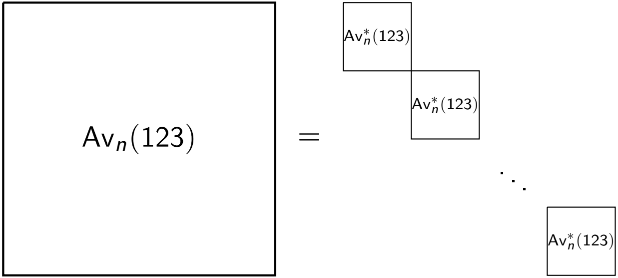
$|\operatorname{Av}_n^*(123)| = c_{n-1}$.

Indecomposable Permutations

Fact

$|\operatorname{Av}_n^*(123)| = c_{n-1}$.

Proof.



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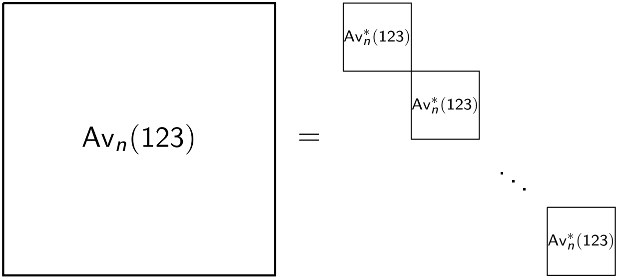


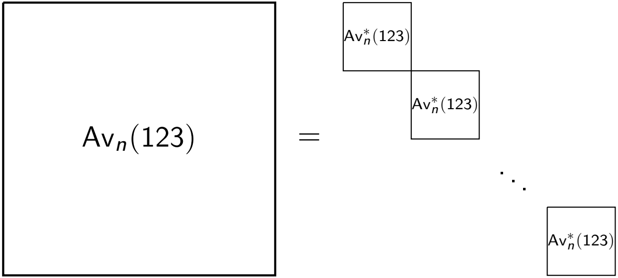
$$C(x) = 1 + C^*(x) + C^*(x)^2 + \ldots = \frac{1}{1 - C^*(x)}$$

Indecomposable Permutations

Fact

$|\operatorname{Av}_n^*(123)| = c_{n-1}.$

Proof.



$$C(x) = 1 + C^*(x) + C^*(x)^2 + \ldots = \frac{1}{1 - C^*(x)}$$

$$C^*(x) = \frac{C(x) - 1}{C(x)} = \frac{(xC(x)^2 + 1) - 1}{C(x)} = xC(x).$$

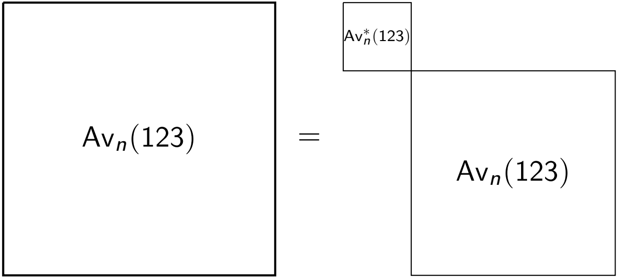
□

Indecomposable Permutations

Fact

$|\operatorname{Av}_n^*(123)| = c_{n-1}$.

Alternate Proof.



$$C(x) = C^*(x)C(x) + 1$$

$$C^*(x) = \frac{C(x) - 1}{C(x)} = \frac{(xC(x)^2 + 1) - 1}{C(x)} = xC(x).$$

□

Solving the System

Conjectures

Solving the System

Conjectures

$$C(x)A(x) = xJ(x)C'(x)$$

$$A^*(x) + B^*(x) = \sum_{n\geq 0} f_{213}(\mathrm{Av}_n^* 132)x^n$$

$$B^*(x)C(x) = 2xB(x)$$

$$A(x) + B(x) = 2\sum_{n\geq 0}\Big(f_{213}(\mathrm{Av}_n^* 132) + f_{231}(\mathrm{Av}_n^* 132)\Big)x^n$$

$$A(x) + B(x) = xB^*(x)$$

$$J^*(x) = 2A^*(x)$$

Solving the System

Corollary

$$C(x)A(x) = xJ(x)C'(x)$$

$$A^*(x) + B^*(x) = \sum_{n\geq 0} f_{213}\big(\operatorname{Av}^*_n 132\big)x^n$$

$$B^*(x)C(x) = 2xB(x)$$

$$A(x) + B(x) = 2\sum_{n\geq 0}\Big(f_{213}\big(\operatorname{Av}^*_n 132\big) + f_{231}\big(\operatorname{Av}^*_n 132\big)\Big)x^n$$

$$A(x) + B(x) = xB^*(x)$$

$$J^*(x) = 2A^*(x)$$

The Lemma

The Lemma

Lemma

$$A^*(x) = \frac{x^3 C(x)}{(1-4x)^{3/2}}$$

Sketch of proof

Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.

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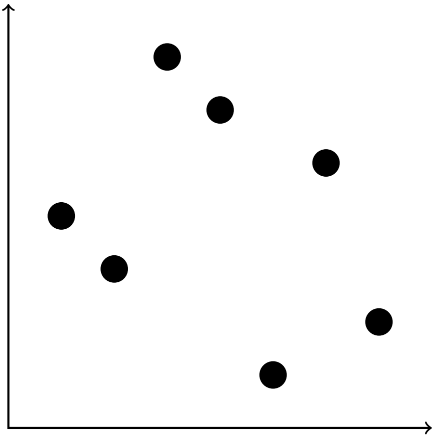
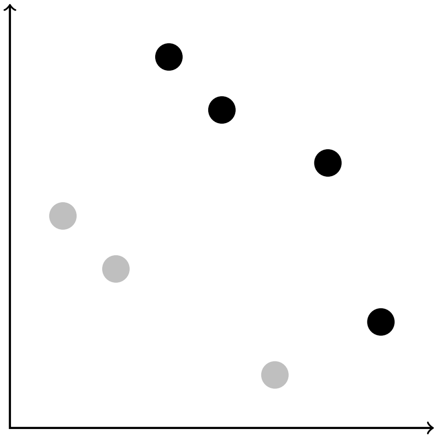


Sketch of proof

Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.



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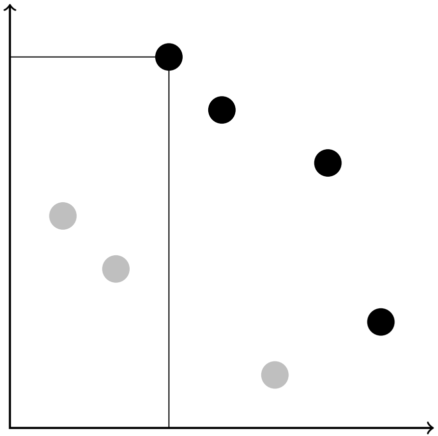

$$f_{213}(p) = \binom{2}{2}$$

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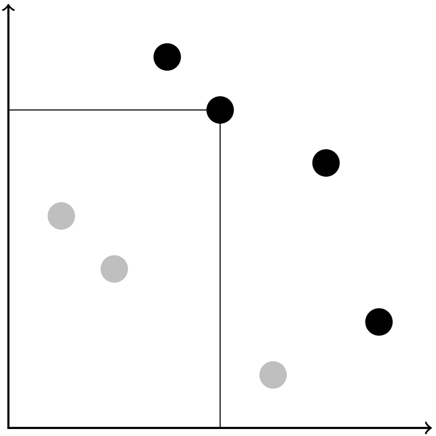

$$f_{213}(p) = \binom{2}{2} + \binom{2}{2}$$

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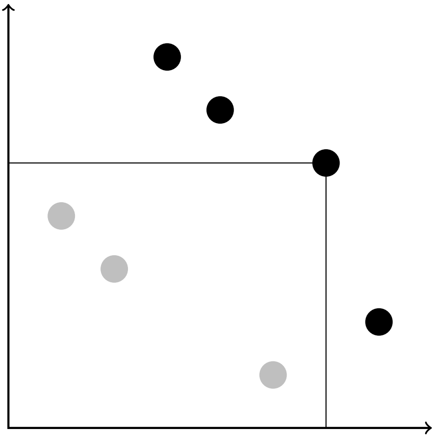

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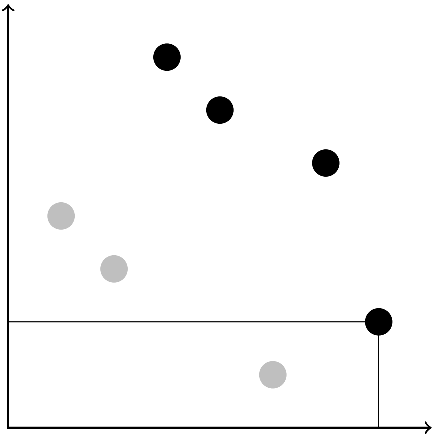

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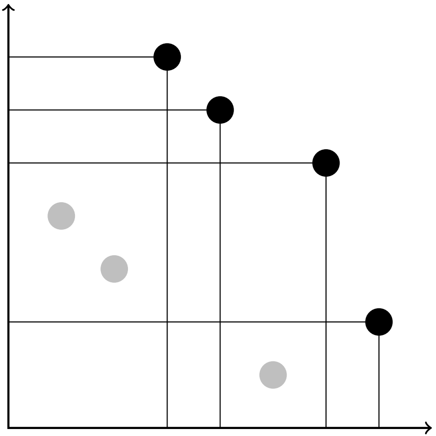

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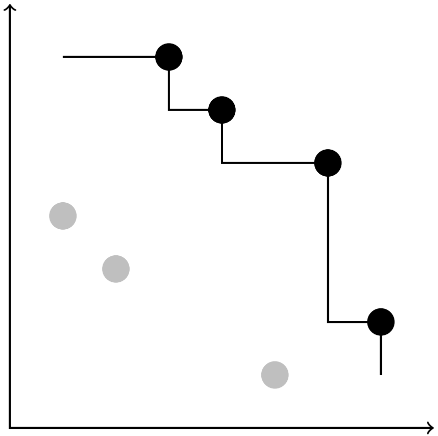

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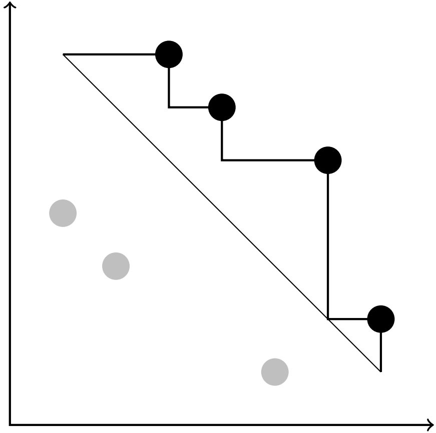

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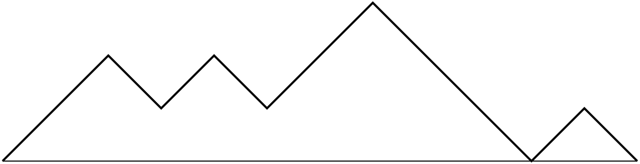

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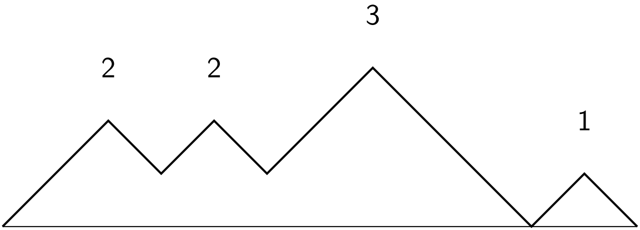

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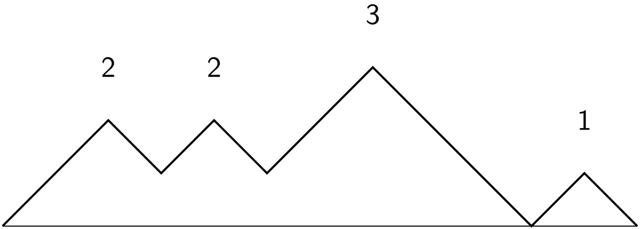
Sketch of proof

Let $h_{n,k}$ denote the total number of peaks at height k in all Dyck paths of semilength n. Let $H(x,u) = \sum_{n,k\geq 0} h_{n,k} x^n u^k$.



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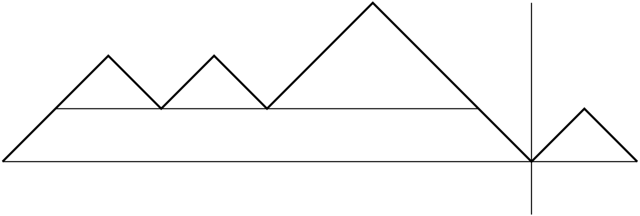

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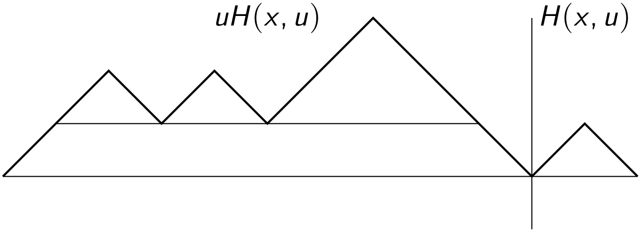

$$H(x,u) = ux(H(x,u)+1)C(x) + xC(x)H(x,u)$$

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□

Corollaries

Corollaries

Corollary

$$f_{231}(\mathsf{Av}_n\,123) = f_{231}(\mathsf{Av}_n\,132)$$

Corollaries

$$A(x) = \frac{x-1}{2(1-4x)} - \frac{3x-1}{2(1-4x)^{3/2}}$$

$$B(x) = \frac{3x-1}{(1-4x)^2} - \frac{4x^2-5x+1}{(1-4x)^{5/2}}$$

$$D(x) = \frac{8x^3-20x^2+8x-1}{(1-4x)^2} - \frac{36x^3-34x^2+10x-1}{(1-4x)^{5/2}}$$

Corollaries

$$a_n = \frac{n+2}{4}\binom{2n}{n} - 3 \cdot 2^{2n-3}$$

$$b_n = (2n-1)\binom{2n-3}{n-2} - (2n+1)\binom{2n-1}{n-1} + (n+4) \cdot 2^{2n-3}$$

$$\begin{aligned} d_n = {} & \frac{1}{6}\binom{2n+5}{n+1}\binom{n+4}{2} - \frac{5}{3}\binom{2n+3}{n}\binom{n+3}{2} \\ & + \frac{17}{3}\binom{2n+1}{n-1}\binom{n+2}{2} - 6\binom{2n-1}{n-2}\binom{n+1}{2} - (n+1) \cdot 4^{n-1}. \end{aligned}$$

Corollaries

$$a_n \sim \sqrt{\frac{n}{\pi}} 4^n$$

$$b_n \sim \frac{n}{2} 4^n$$

$$d_n \sim \frac{8}{3} \sqrt{\frac{n^3}{\pi}} 4^n.$$

Larger patterns

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Lemma

$$\begin{aligned} 2A(x) \quad &+2B(x) \quad +D(x) &= \quad &\frac{x^3}{6}(C(x))''' \\ 4A(x) \quad &+2B(x) &= \quad &x^3(J(x)/x^2)' \end{aligned}$$

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Theorem

For large enough n, the descending pattern of length k occurs more often than any other length k pattern in $\mathrm{Av}_n(123)$.

Further Directions

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Question

Are there any other ‘surprising’ symmetries across or within permutation classes?

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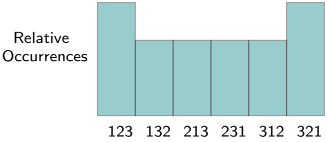


Av 2413

The Pattern Poset

The Pattern Poset

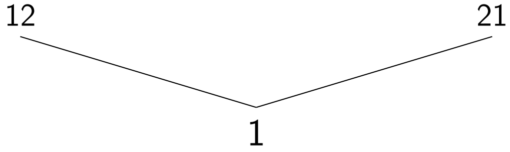
1

12

21

1

The Pattern Poset



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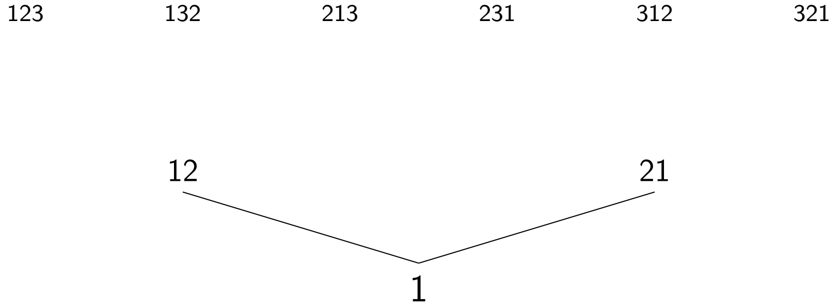

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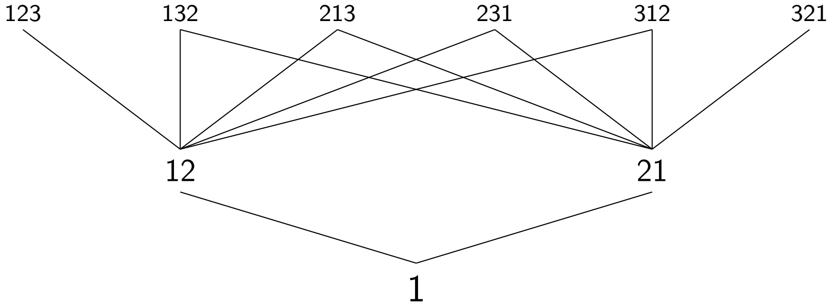

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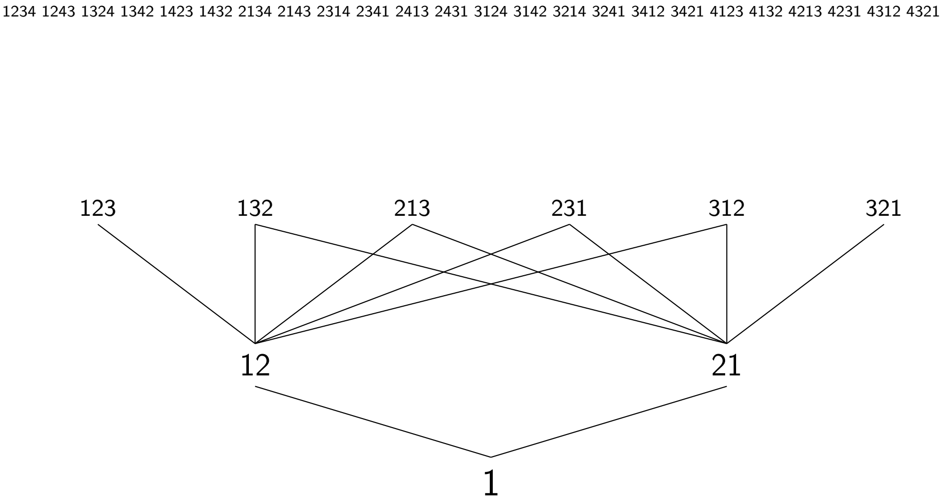

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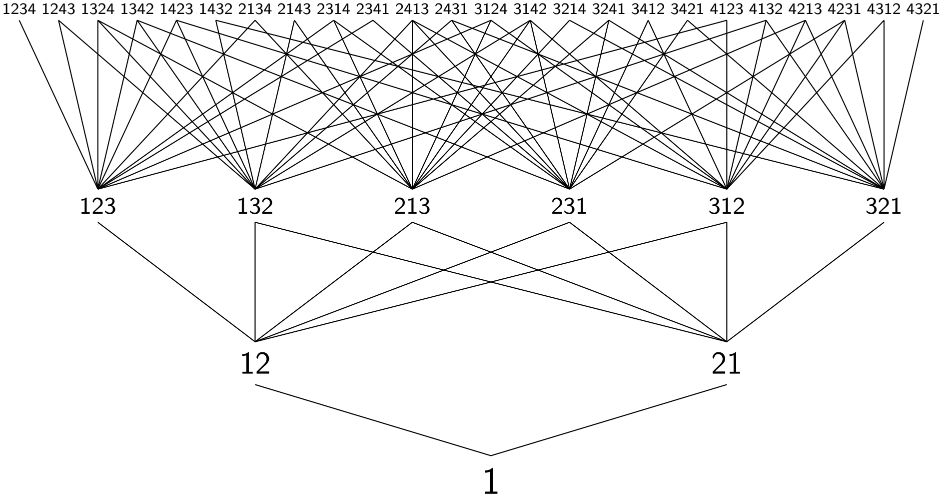

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