Real Applications of Structural Combinatorics
Three Case Studies

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CCICADA

March 13th, 2014

Patterns in Data

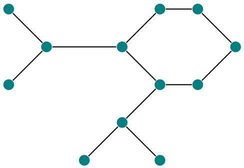
Genome Rearrangement

Combinatorial Testing

Relational Structures

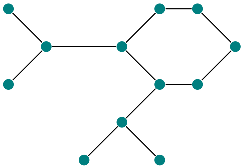
Combinatorial Structures

Combinatorial Structures



Graphs

Combinatorial Structures



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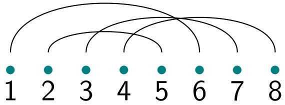
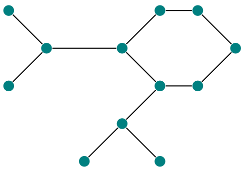


Matchings

Combinatorial Structures



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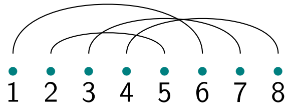

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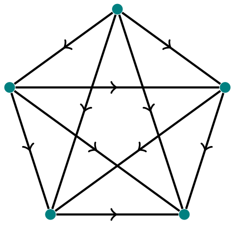

Tournaments

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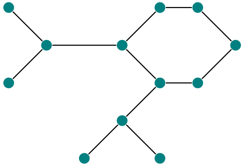

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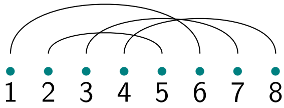

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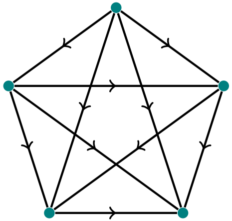

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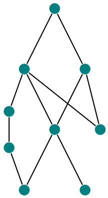

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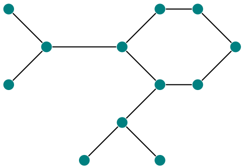

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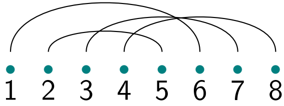

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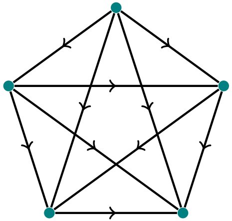

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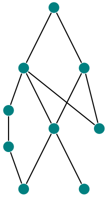

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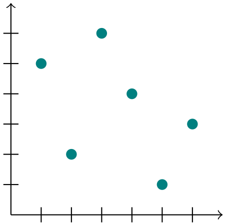

Permutations

Relational Structures

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A *relational structure* consists of a *ground set* together with a set of *relations*, describable using first-order logic.

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Example

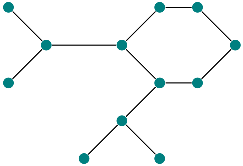
A graph is a ground set (vertices) together with a 2-relation (edges) which is symmetric and nonreflexive.
That is:

$$\mathcal{G} = (\mathcal{S}, \mathcal{R}), \text{ where } \mathcal{R} \subset \mathcal{S} \times \mathcal{S},$$

with

$$(x, y) \in \mathcal{R} \implies (y, x) \in \mathcal{R}, \text{ and } (x, x) \notin \mathcal{R} \ \forall x \in \mathcal{S}.$$

Combinatorial Structures



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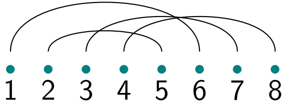

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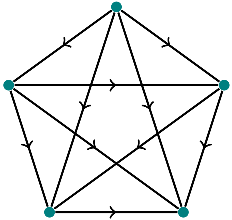

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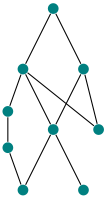

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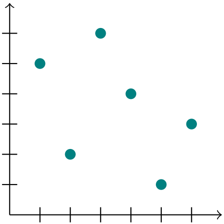

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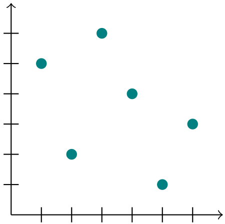
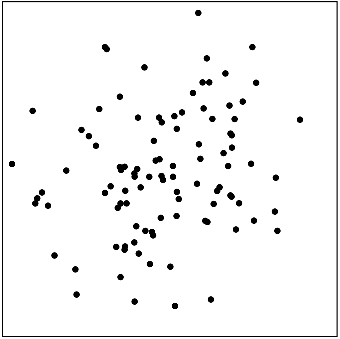


Permutations

Patterns in Data

(and Permutations)

Random Data



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Definition

An *permutation of length n* is a bijection from the set $[n] = \{1, 2, \ldots n\}$ to itself. The *one-line notation* for a permutation π is

$$\pi = \pi(1)\pi(2)\ldots\pi(n).$$

The set of all permutations of length n is denoted $\mathfrak{S}_n$.

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- The six permutations of length 3 are

$$\mathfrak{S}_3 = \{123, 132, 213, 231, 312, 321\}.$$

Plotting Permutations

Definition

If π is a permutation of length n, then the *plot* of π is the set of points

$$\{(1, \pi(1)), (2, \pi(2)), \cdots (n, \pi(n))\} \subset \mathbb{R}^2$$

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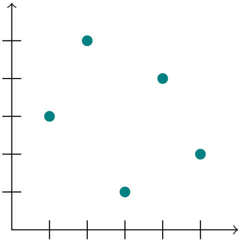

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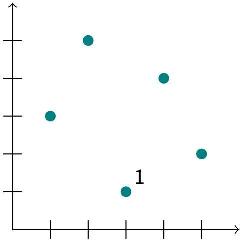

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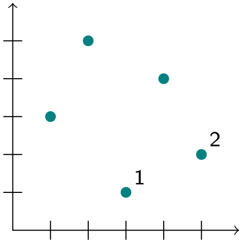

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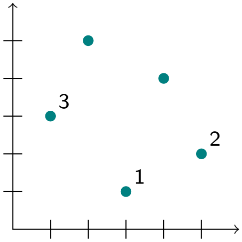

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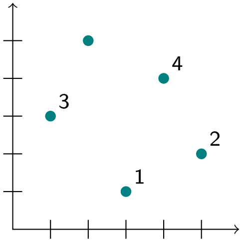

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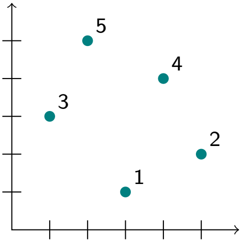

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Dots on a Plane

Definition

Let A and B be two sets of n points in $\mathbb{R}^2$, each with the property that no two points lie on the same horizontal or vertical line.
Say that A is *order isomorphic* to B (denoted $A \sim B$) if A can be transformed into B by stretching, contracting, and translating the axes horizontally and vertically.

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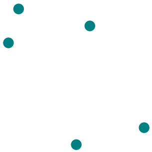

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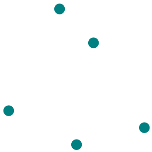

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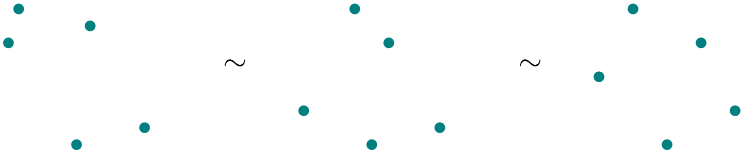

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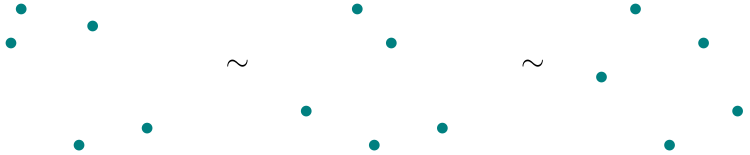

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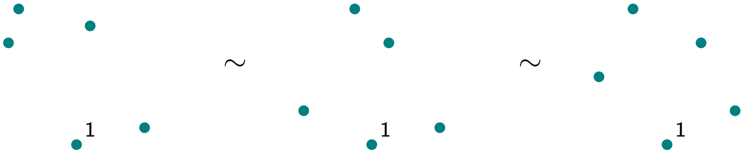

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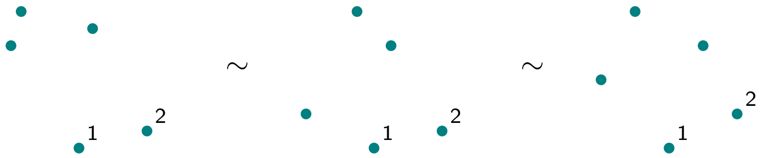

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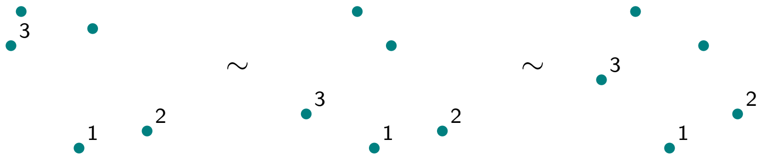

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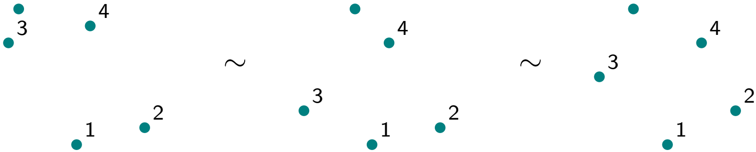

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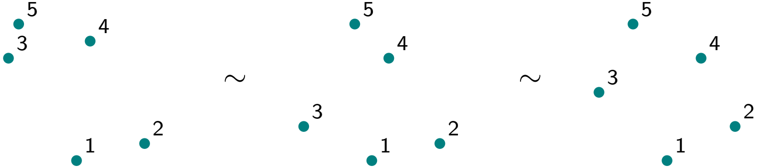

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Permutation Symmetries

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For a permutation $\pi = \pi_1\pi_2\ldots\pi_n$, the reverse, the complement, and the inverse of π are denoted π^r, π^c, and π^{-1}, and defined as follows:

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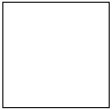

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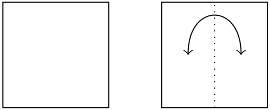

π π^r

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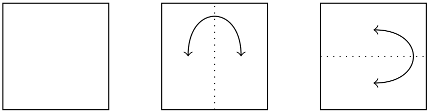

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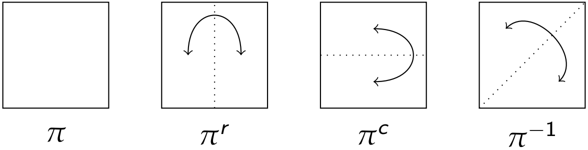
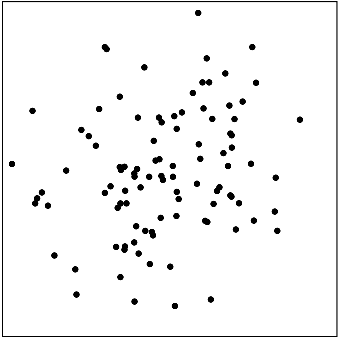


Random Data



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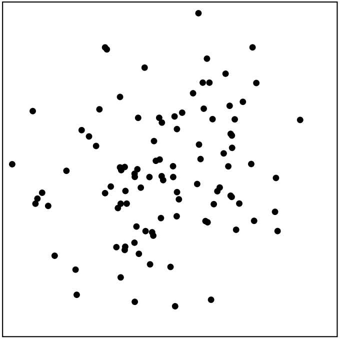

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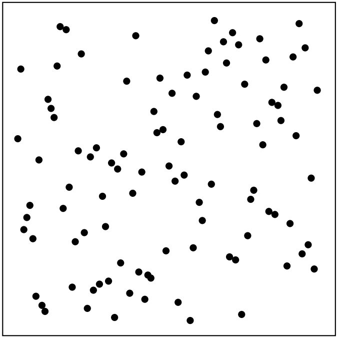

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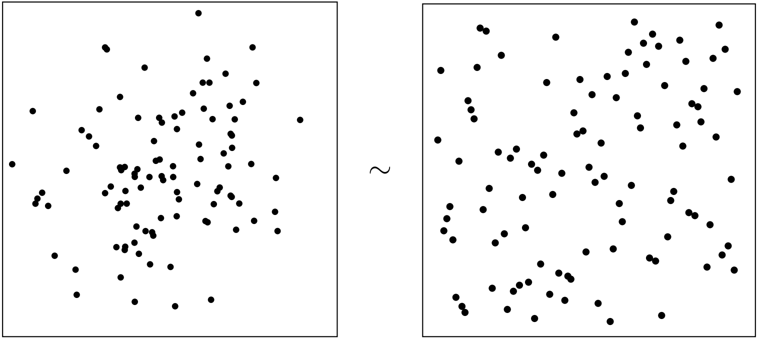

$\pi =$ 61 84 31 35 39 28 9 54 6 4 74 71 68 85 98 38 97 45 12 27 57 89 30 5 55 11 58
13 42 32 14 53 2 51 20 56 80 10 43 95 17 50 8 16 15 70 63 81 64 24 52 76 47
7 60 49 82 1 25 75 40 34 83 90 46 100 69 65 93 86 22 96 21 92 3 79 29 41
44 66 94 59 87 37 73 36 72 67 78 19 33 88 62 99 23 91 26 48 18 77

Permutation Patterns

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Definition

Let $\pi = \pi(1)\pi(2)\cdots\pi(n)$ and $\sigma = \sigma(1)\sigma(2)\cdots\sigma(k)$ be two permutations. π *contains* σ *as a pattern* (written $\sigma \prec \pi$) if there is some subsequence $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ which is order isomorphic to the entries of σ (i.e., $\pi(i_j) < \pi(i_k)$ if and only if $\sigma(j) < \sigma(k)$).

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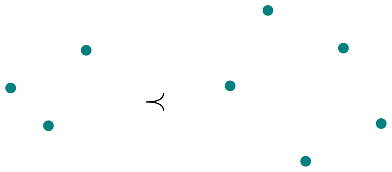
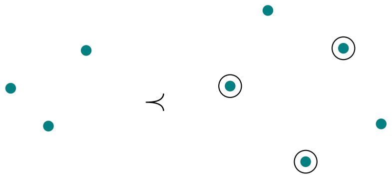


$$213 \quad \prec \quad 35142$$

Permutation Patterns

Definition

Let $\pi = \pi(1)\pi(2)\cdots\pi(n)$ and $\sigma = \sigma(1)\sigma(2)\cdots\sigma(k)$ be two permutations. π *contains* σ *as a pattern* (written $\sigma \prec \pi$) if there is some subsequence $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ which is order isomorphic to the entries of σ (i.e., $\pi(i_j) < \pi(i_k)$ if and only if $\sigma(j) < \sigma(k)$).



$213 \quad \prec \quad \mathbf{3}5\mathbf{14}2$

Permutation Patters

Example

The pattern 12 is contained in all permutations *except* for the decreasing ones:

$$12 \not\preceq n \ldots 321.$$

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Definition

If a permutation π does not contain a pattern σ, we say that π *avoids* σ. The set of all permutations which avoid a given pattern (or set of patterns) σ is denoted

$$\mathrm{Av}(\sigma).$$

Permutation Classes

Permutation Classes

Definition

A *permutation class* is a set $\mathcal{C}$ of permutations for which, if $\pi \in \mathcal{C}$ and $\sigma \prec \pi$, then $\sigma \in \mathcal{C}$. Let $\mathcal{C}_n$ denote the set of permutations of length n in $\mathcal{C}$.

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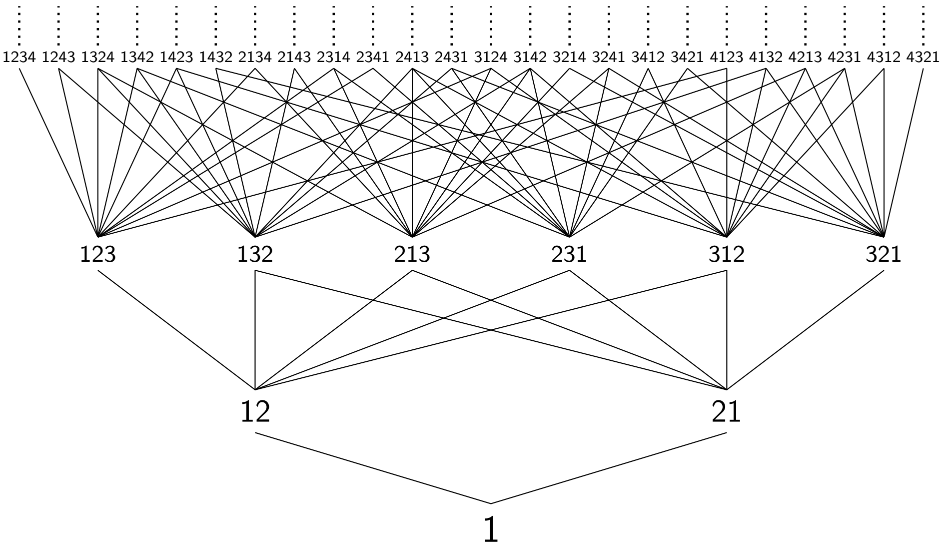
Theorem (Marcus and Tardos, 2004)

Every proper permutation class has a finite exponential growth rate. That is, for any proper class $\mathcal{C}$, there exists a real number s such that

$$\limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|} = s.$$

This number s is the *growth rate* of the class.

Permutation Classes - Growth Rates



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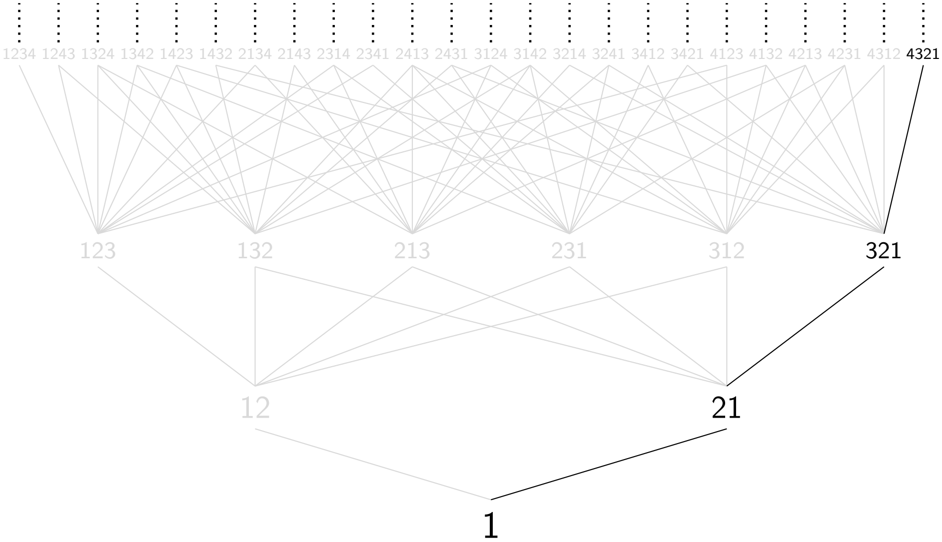

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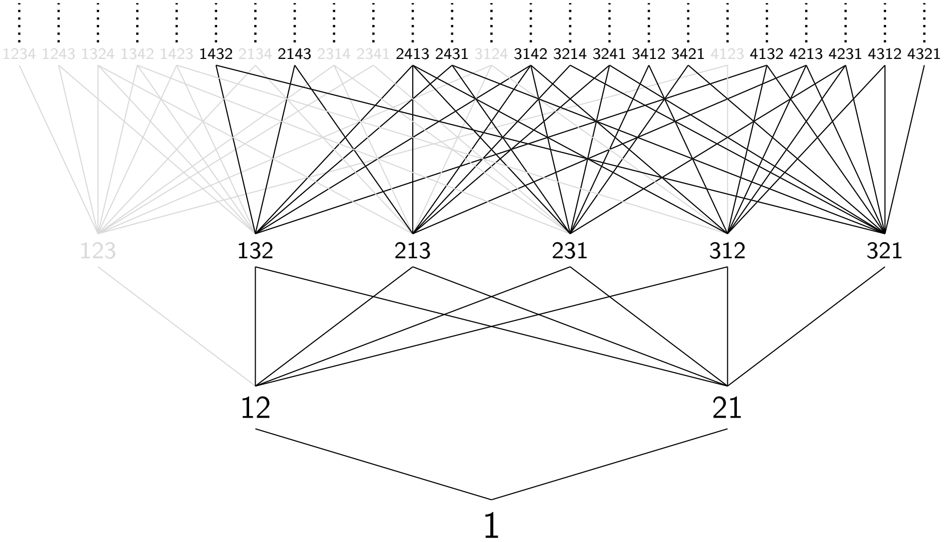

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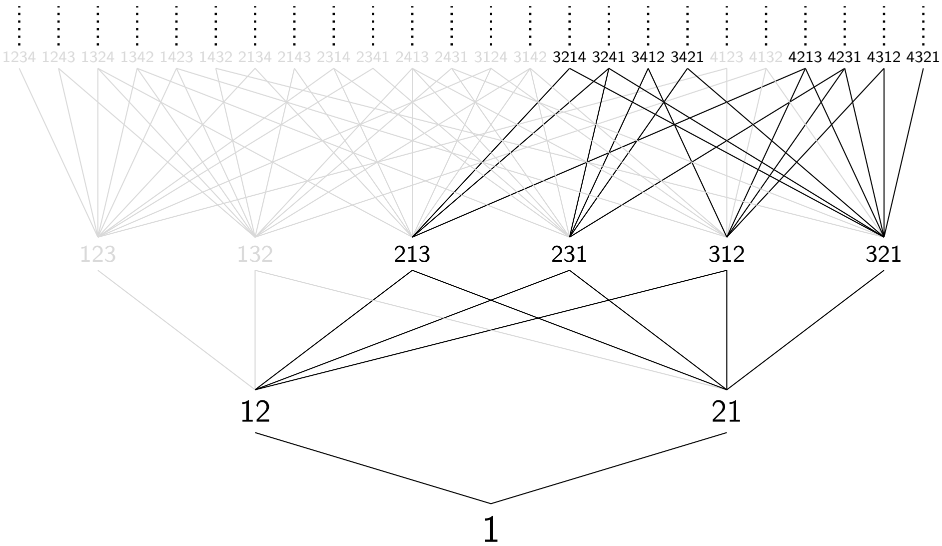

The Class Av(132)

Definition

Let c_n be the number of permutations of length n which *avoid* the pattern 132, and $C(x) = \sum_{n \geq 0} c_n x^n$.

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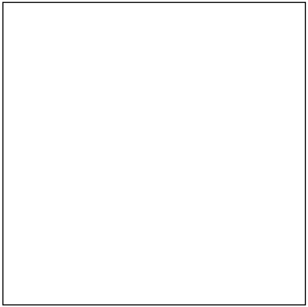

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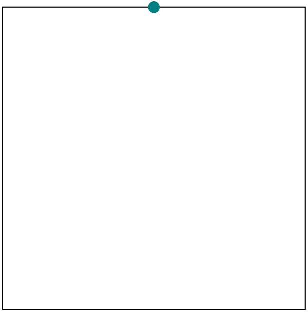

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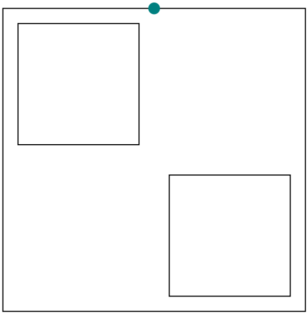

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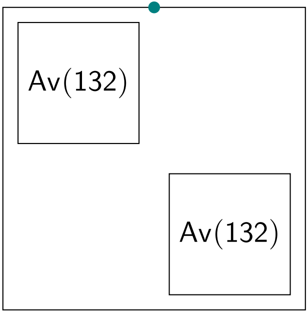

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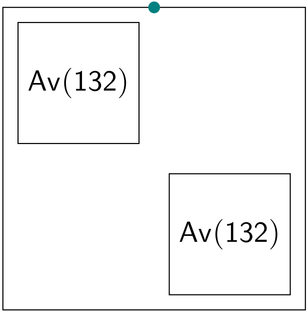

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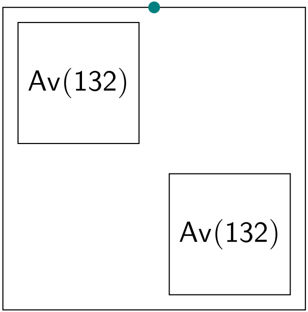

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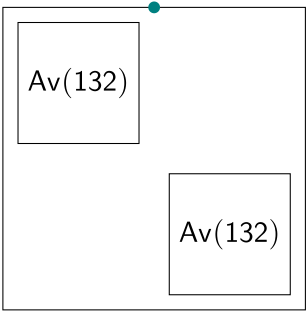

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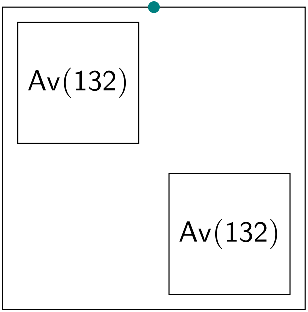

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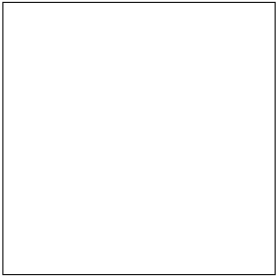

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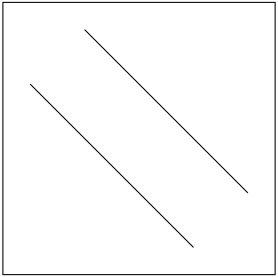

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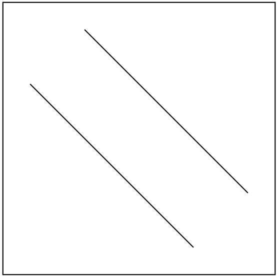

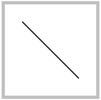

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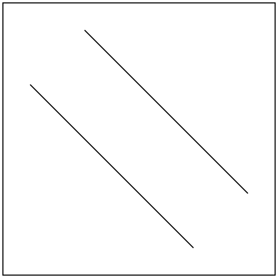

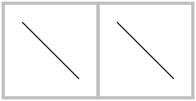

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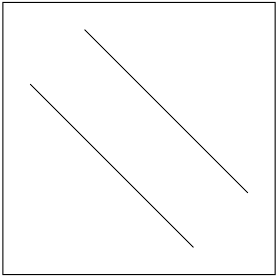

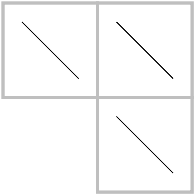

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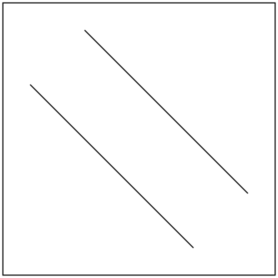

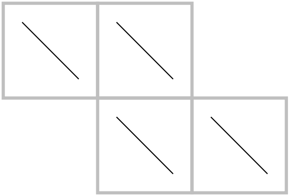

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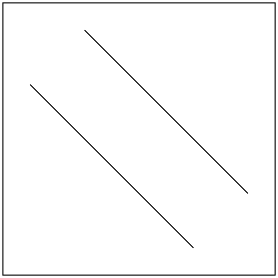

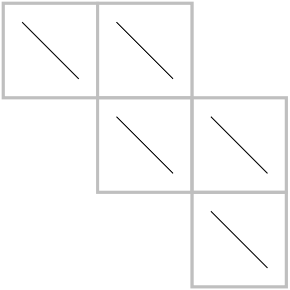

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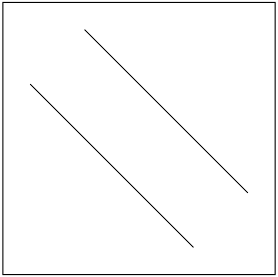

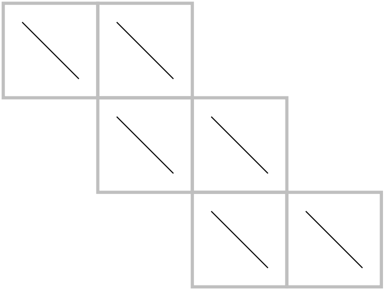

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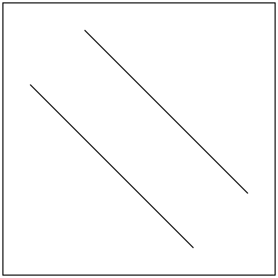
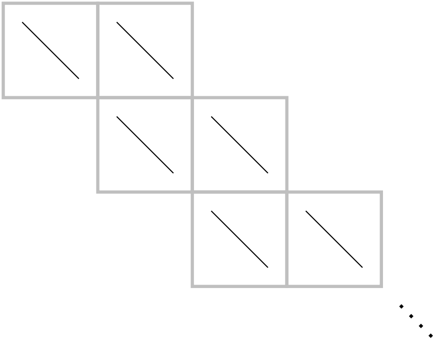
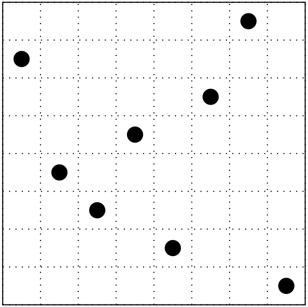




Av(132) and Av(123)



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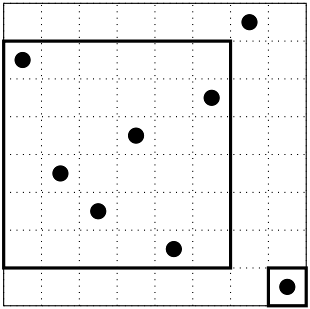

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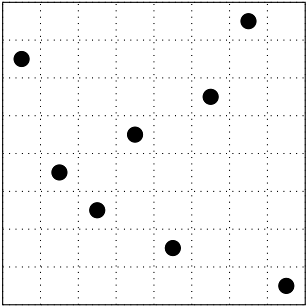

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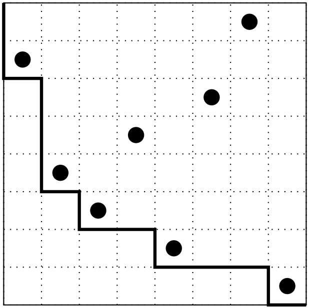

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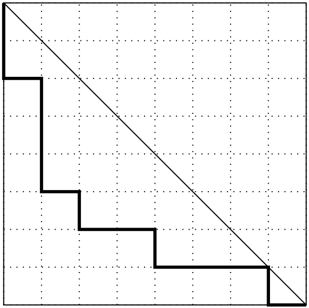

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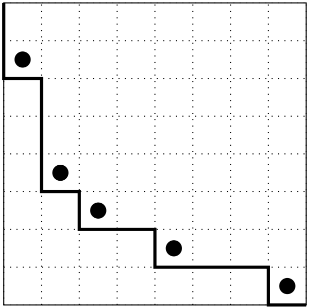

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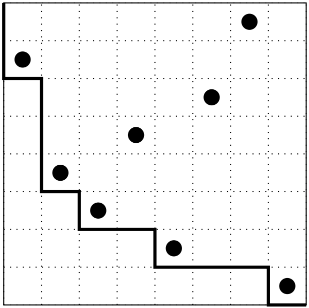

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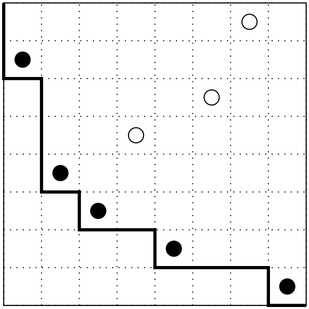

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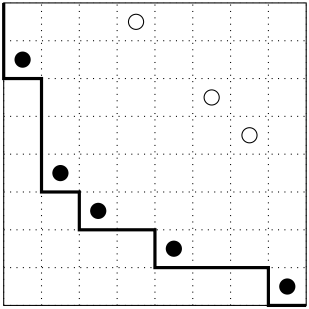
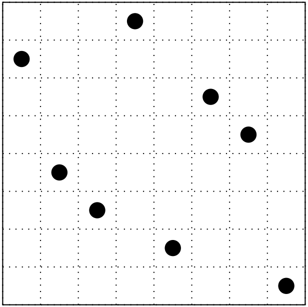


$\mathsf{Av}(132) \mapsto \mathsf{Av}(123)$

Av(132) and Av(123)



Av(123)

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$$|\,\mathsf{Av}_n(123)| = |\,\mathsf{Av}_n(132)|$$

Av(132) and Av(123)

$$|\operatorname{Av}_n(123)| = |\operatorname{Av}_n(132)| = \frac{1}{n+1}\binom{2n}{n}.$$

Pattern Occurrences

Patterns

Say that one permutation π contains another permutation σ *as a pattern* (denoted $\sigma \prec \pi$) if the plot of π contains a subset which is equivalent to the plot of σ. The number of occurrences of σ in π (denoted $\nu_\sigma(\pi)$) is the number of such subsets.

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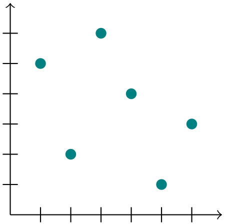

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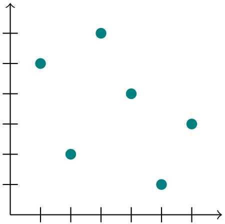

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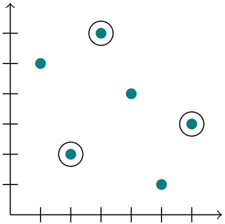

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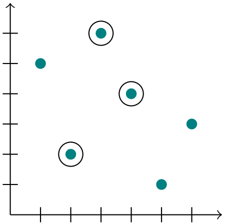

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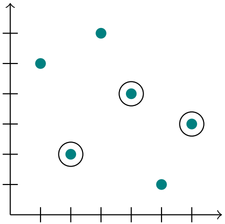

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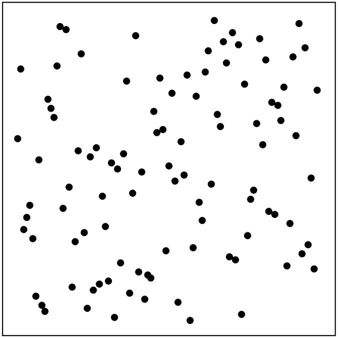
Random Data



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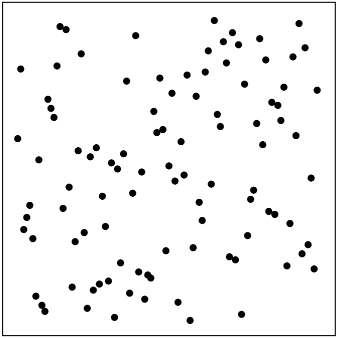

ν_{12}	ν_{21}	Avg
2803	2147	2475

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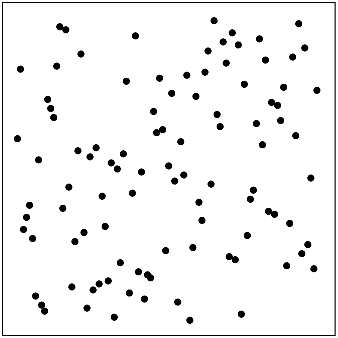

ν_{12}	ν_{21}	Avg
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ν_{123}	ν_{132}	ν_{213}	ν_{231}	ν_{312}	ν_{321}	Avg
35357	30063	31414	22321	23348	19197	26950

Patterns as Random Variables

Theorem (Bóna 2007)

For a (uniformly) randomly selected permutation of length n, the random variables ν_σ are asymptotically normal as n approaches infinity.

Theorem (Janson, Nakamura, Zeilberger 2013)

For a randomly selected permutation of length n and two patterns σ and ρ, the random variables ν_σ and ν_ρ are asymptotically jointly normally distributed as $n \to \infty$.

Motivation

Fact

In $\mathfrak{S}_n$, the number of occurrences of a specific pattern depends only on the length of the pattern. That is, for a pattern $\sigma \in \mathfrak{S}_k$, we have

$$\nu_\sigma(\mathfrak{S}_n) = \frac{n!}{k!}\binom{n}{k}.$$

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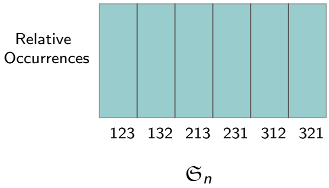

Motivation

Fact

In $\mathfrak{S}_n$, the number of occurrences of a specific pattern depends only on the length of the pattern. That is, for a pattern $\sigma \in \mathfrak{S}_k$, we have

$$\nu_\sigma(\mathfrak{S}_n) = \frac{n!}{k!}\binom{n}{k}.$$

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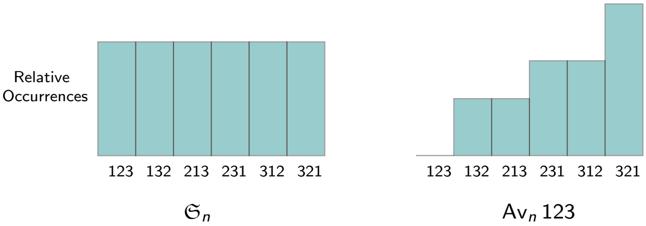

Previous Results

Theorem (Bóna)

In $\mathrm{Av}_n 132$, the pattern 123 is the least common, 321 is the most common, and $\nu_{213} = \nu_{231} = \nu_{312}$.

Data

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
4	10	0	11	11	11	13
5	68	0	81	81	81	109
6	392	0	500	500	500	748
7	2063	0	2794	2794	2794	4570

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
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Av 123

length	123	132	213	231	312	321
3	0	1	1	1	1	1

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
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Av 123

length	123	132	213	231	312	321
3	0	1	1	1	1	1
4	0	9	9	11	11	16

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
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Av 123

length	123	132	213	231	312	321
3	0	1	1	1	1	1
4	0	9	9	11	11	16
5	0	57	57	81	81	144
6	0	312	312	500	500	1016
7	0	1578	1578	2794	2794	6271

Patterns Within Av(123)

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Theorem

The total nuber of 231 (and 312) patterns is identical within the sets $\mathsf{Av}_n(123)$ and $\mathsf{Av}_n(132)$.

Patterns Within Av(123)

Theorem

The total nuber of 231 (and 312) patterns is identical within the sets $\mathrm{Av}_n(123)$ and $\mathrm{Av}_n(132)$.
Further, within $\mathrm{Av}_n(123)$,

$$\nu_{132} = \nu_{213} \sim \sqrt{\frac{n}{\pi}} 4^n,$$
$$\nu_{231} = \nu_{312} \sim \frac{n}{2} 4^n,$$
$$\text{and} \quad \nu_{321} \sim \frac{8}{3} \sqrt{\frac{n^3}{\pi}} 4^n.$$

Sketch of Proof: Patterns in Av(123)

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ν_{132} ν_{213} ν_{231} ν_{312} ν_{321}

Sketch of Proof: Patterns in Av(123)

$$\nu_{132} + \nu_{213} + \nu_{231} + \nu_{312} + \nu_{321} = \binom{n}{3} c_n$$

(Both sides count the number of length three patterns)

Sketch of Proof: Patterns in Av(123)

$$2\nu_{132} + 2\nu_{213} + \nu_{231} + \nu_{312} = (n-2)\nu_{12}$$

(Count triples containing a 12 pattern . . .)

Sketch of Proof: Patterns in Av(123)

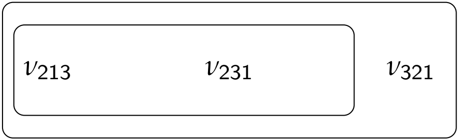
ν_{132} ν_{213} ν_{231} ν_{312} ν_{321}

Sketch of Proof: Patterns in Av(123)

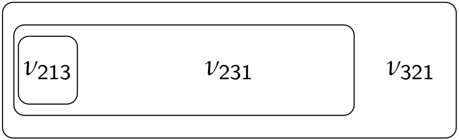
$\nu_{132} = \nu_{213}$ $\nu_{231} = \nu_{312}$ ν_{321}

(Since Av(123) is closed under inversion)

Sketch of Proof: Patterns in Av(123)



Sketch of Proof: Patterns in Av(123)



Sketch of Proof: Counting 213 Patterns

Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.

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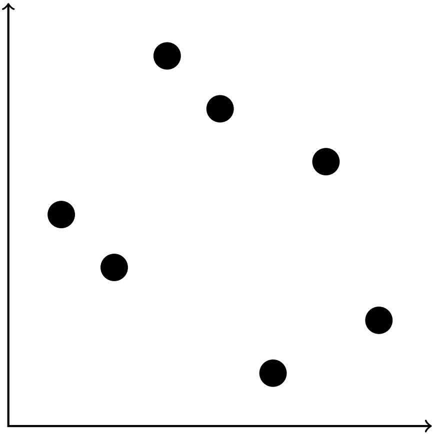

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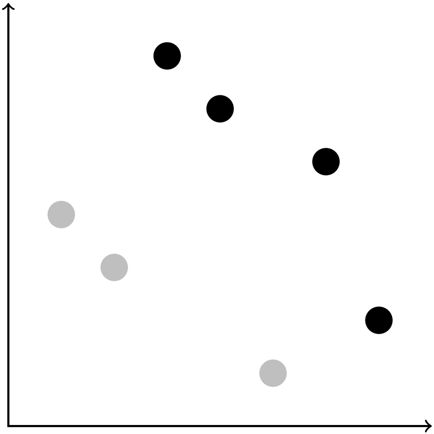

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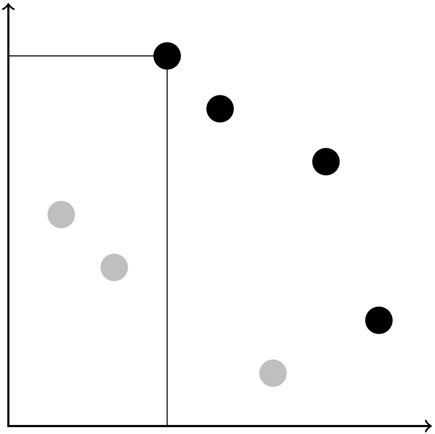

$$\nu_{213}(p) = \binom{2}{2}$$

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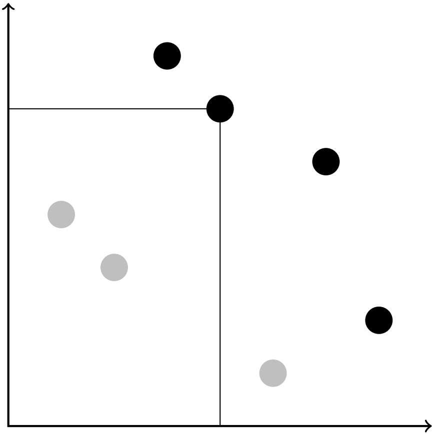

$$\nu_{213}(p) = \binom{2}{2} + \binom{2}{2}$$

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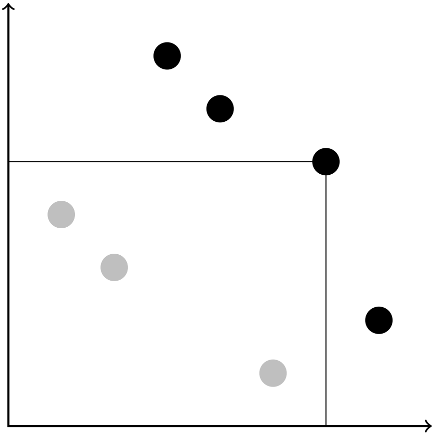

$$\nu_{213}(p) = \binom{2}{2} + \binom{2}{2} + \binom{3}{2}$$

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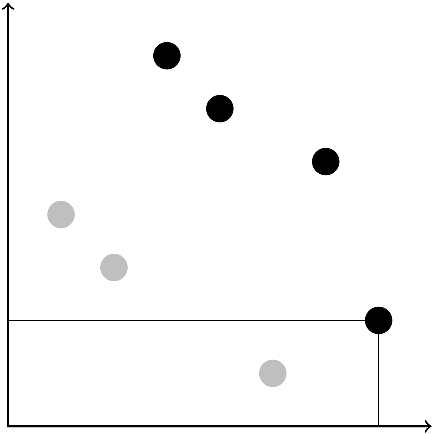

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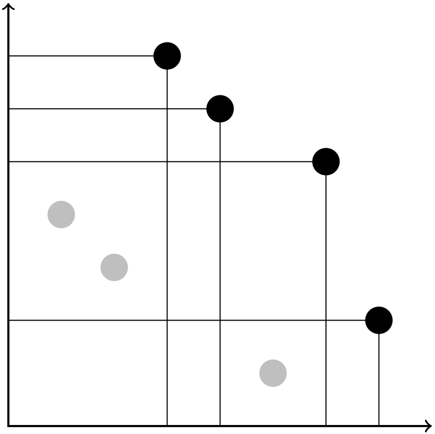

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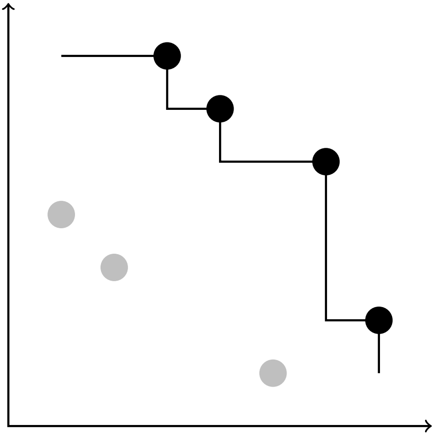

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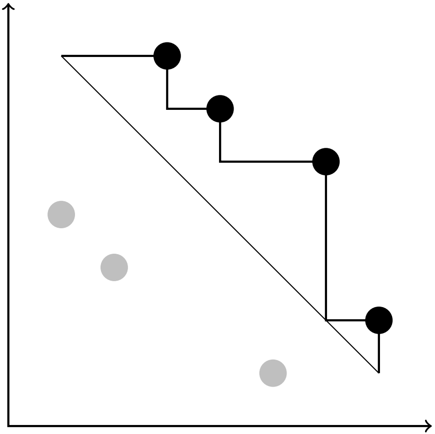

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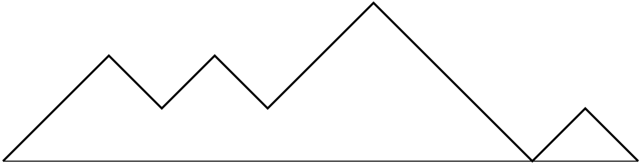

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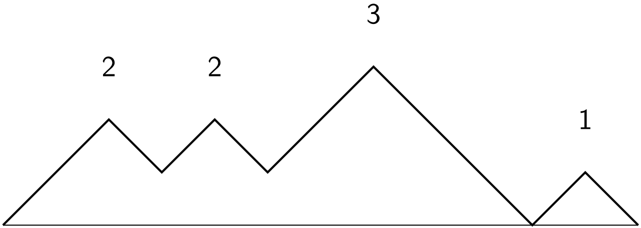

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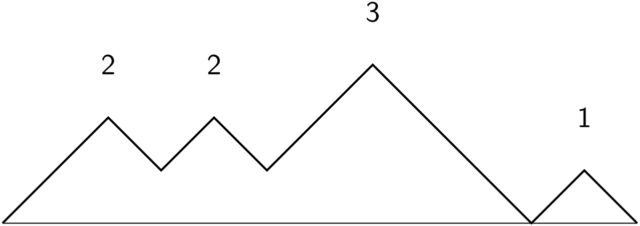
Sketch of Proof: Counting 213 Patterns

Let $h_{n,k}$ denote the total number of peaks at height k in all Dyck paths of semilength n. Let $H(x, u) = \sum_{n,k\geq 0} h_{n,k} x^n u^k$.



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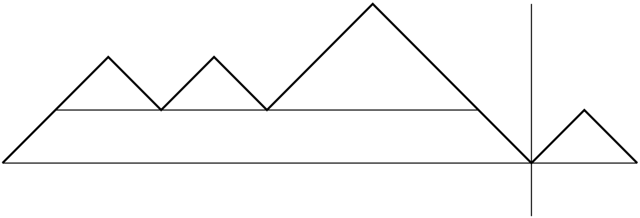

$$H(x,u) =$$

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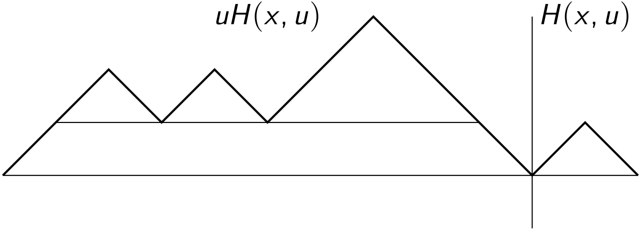

$$H(x,u) = ux(H(x,u)+1)C(x) + xC(x)H(x,u)$$

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$$\sum_{n\geq 0} \nu_{213}(\mathrm{Av}_n^*(123))x^n = \sum_{n\geq 0} \binom{k}{2} h_{n-1,k} x^n$$

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$$= x^3 + 7x^4 + 38x^5 + 187^6 + \ldots$$

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□

Results

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$$\nu_{231}(\mathsf{Av}_n\, 123) = \nu_{231}(\mathsf{Av}_n\, 132)$$

Results

$$\nu_{213} = \frac{n+2}{4}\binom{2n}{n} - 3 \cdot 2^{2n-3}$$

$$\nu_{231} = (2n-1)\binom{2n-3}{n-2} - (2n+1)\binom{2n-1}{n-1} + (n+4) \cdot 2^{2n-3}$$

$$\begin{aligned}\nu_{321} = {} & \frac{1}{6}\binom{2n+5}{n+1}\binom{n+4}{2} - \frac{5}{3}\binom{2n+3}{n}\binom{n+3}{2} \\ & + \frac{17}{3}\binom{2n+1}{n-1}\binom{n+2}{2} - 6\binom{2n-1}{n-2}\binom{n+1}{2} - (n+1) \cdot 4^{n-1}.\end{aligned}$$

Further Directions/Applications

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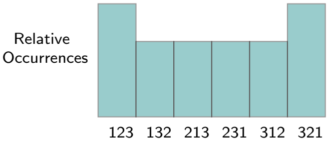

Av 2413

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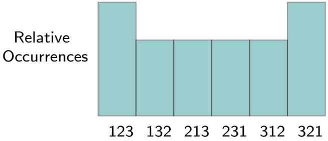

Av 2413

Question

What about multiset permutations?

Genome Rearrangement

Evolutionary Distance

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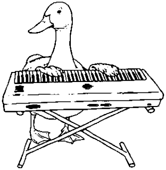

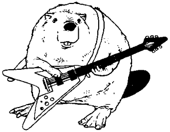

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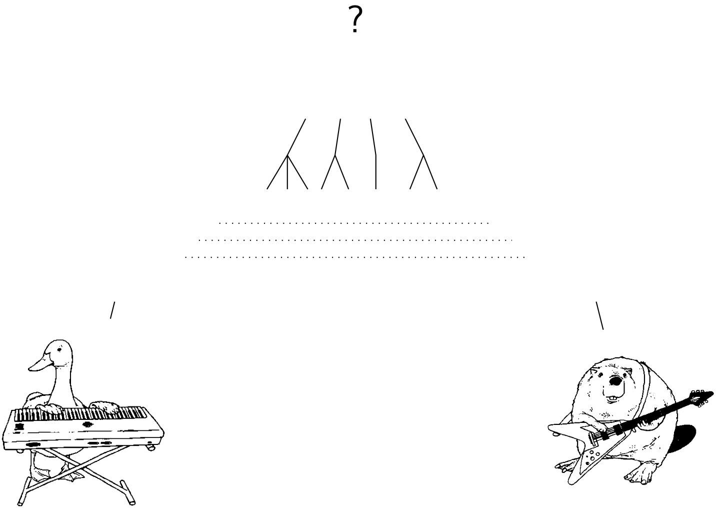

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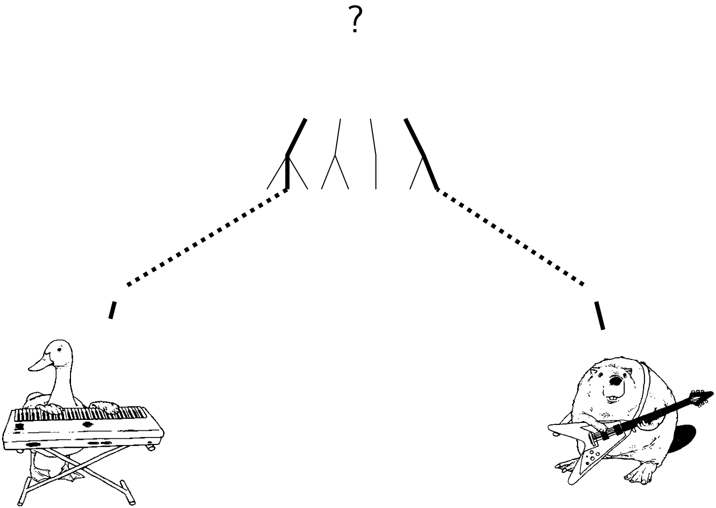

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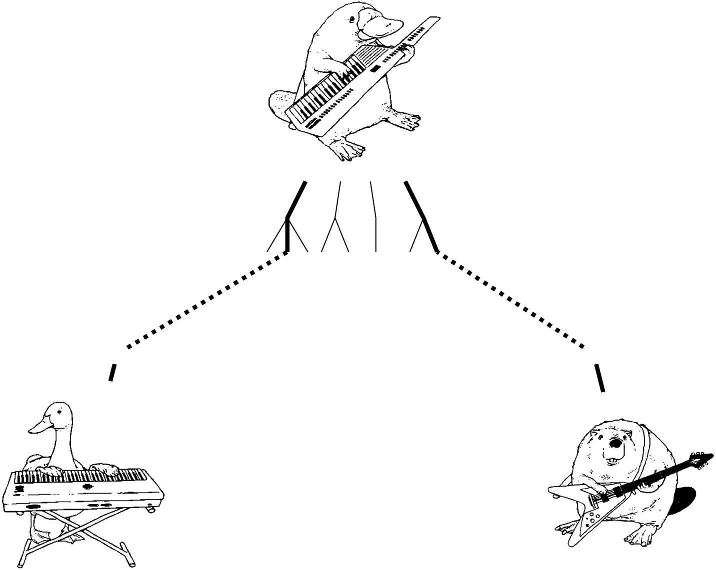

Genome Rearrangement

CCGTGCTACACTGCGATCCGACT . . .

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CCG *TGCTA* *CACTGCGATCCGACT* . . .

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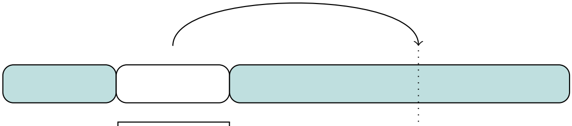

CCG *TGCTA* *CACTGCGATCCGACT* ...

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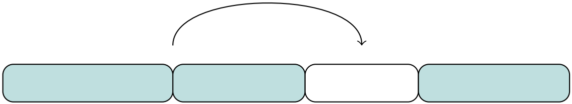
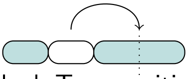


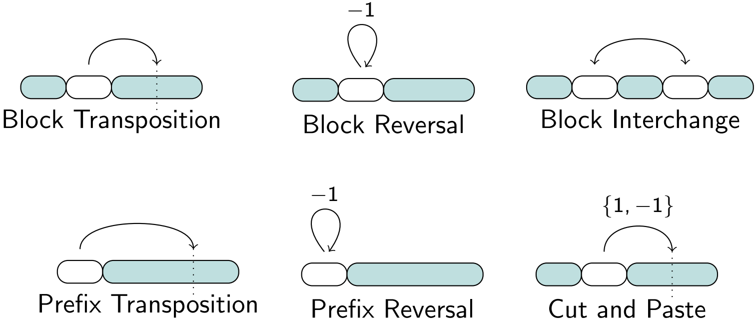
CCGCACTGCGATC *TGCTA* *CGACT* ...

Block Transformations



Block Transposition

Block Transformations



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For a given block transformation, how many mutations does it take to turn one genome into another?

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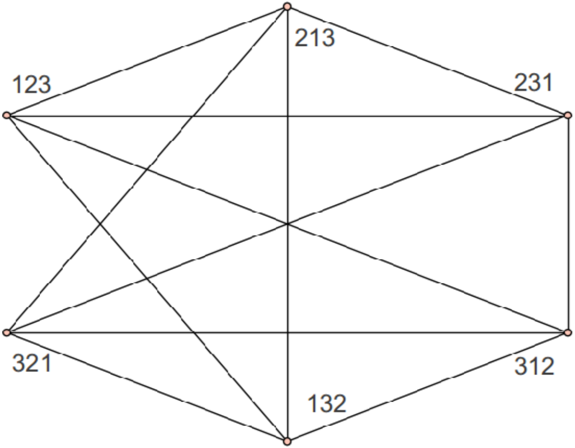
How many block transformations does it take to turn one permutation into another?

Question

Given the increasing permutation, how many block transformations does it take to sort it into a given permutation?

Example: Block Transpositions



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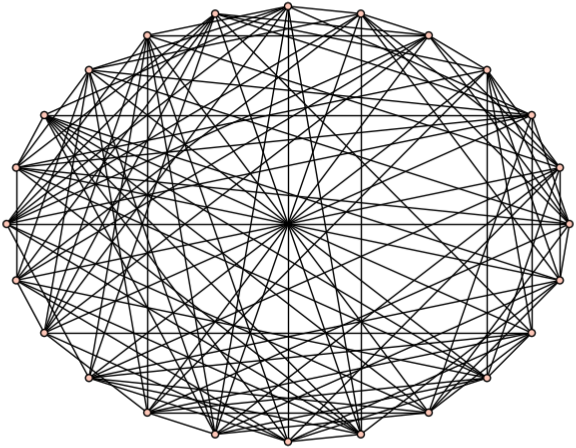

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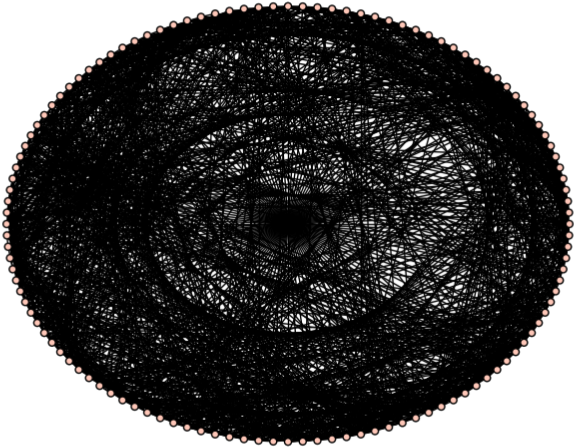

Block Transformations and Polynomial Classes

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Definition

A *polynomial class* is a class $\mathcal{C}$ for which $f(n) = |\mathcal{C}_n|$ is given by a polynomial for large enough n.

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Theorem

For a given block transformation and a positive integer k, the set of permutations which are at most k transformations from the permutation $123\ldots n$ forms a polynomial class.

Peg Figures

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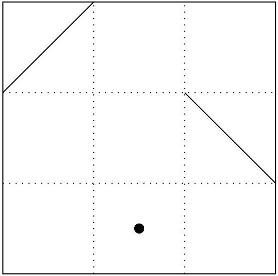
Definition

A *peg figure* consists of a grid of increasing and decreasing lines and dots, with one non-empty cell per row and column:



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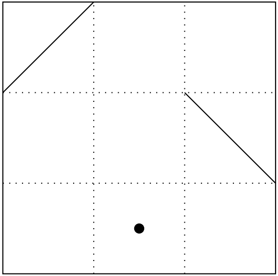

Definition

A *peg permutation* is a permutation $\tilde{\pi}$ with each entry decorated by either a $+$, $-$, or $\bullet$.

Peg Figures

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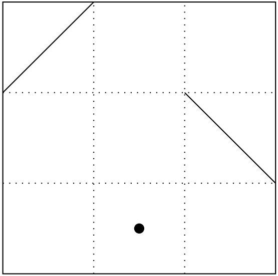

Definition

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$$\tilde{\pi} = \overset{+}{3}\ \overset{\bullet}{1}\ \overset{-}{2}$$

Peg Classes

Definition

Let $\mathcal{C}(\widetilde{\pi})$ denote the class of permutations which can be drawn on the figure corresponding to $\widetilde{\pi}$.

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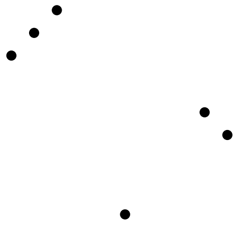

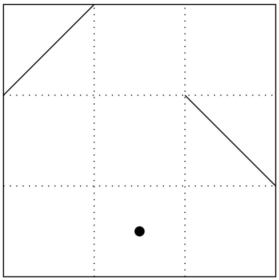

$$456132 \in \mathcal{C}(\overset{+}{3}\overset{\bullet}{1}\overset{-}{2})$$

Polynomial Classes

Theorem (Vatter et. al.)

A permutation class is a polynomial class if and only if it can be expressed as the union of classes of the form $\mathcal{C}(\widetilde{\pi})$.

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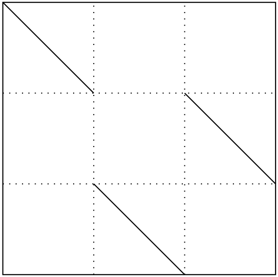
Algorithm

Takes peg permutations as input, and returns the polynomial enumerating the class.

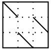
Example: Av(123, 231)

Fact

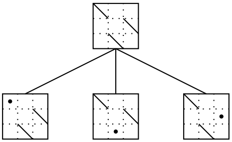
The class Av(123, 231) is equal to the peg class $\mathcal{C}(\bar{3}\bar{1}\bar{2})$.



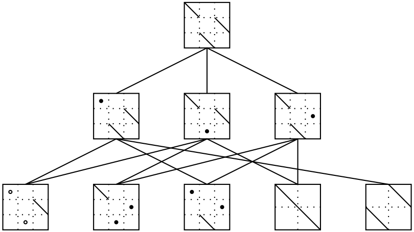
Example: Av(123, 231)



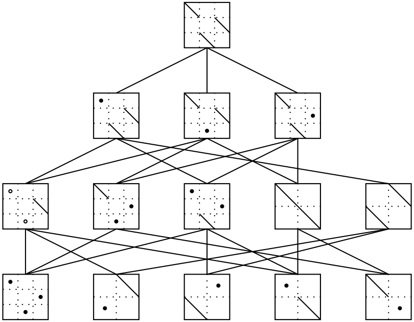
Example: $\mathrm{Av}(123, 231)$



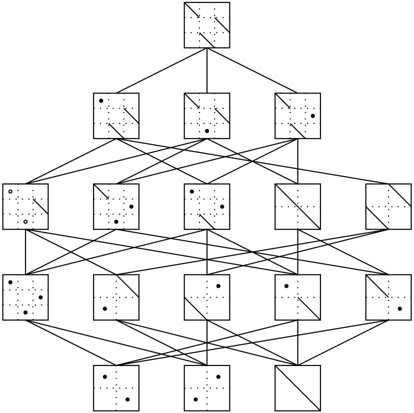
Example: $\text{Av}(123, 231)$



Example: Av(123, 231)



Example: Av(123, 231)



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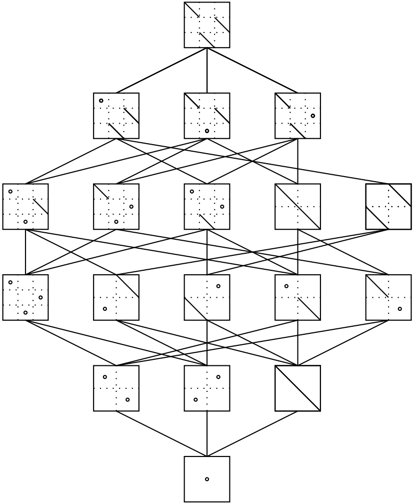

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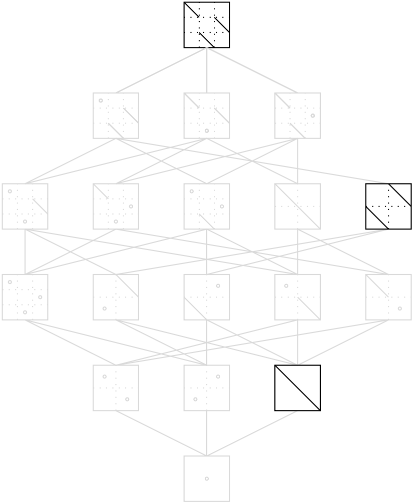

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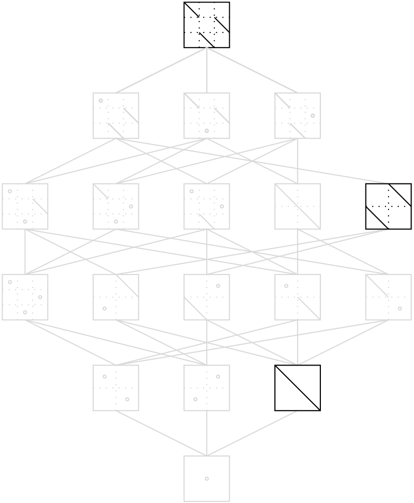

$$\frac{z^3}{(1-z)^3}$$

$$+$$

$$\frac{z^2}{(1-z)^2}$$

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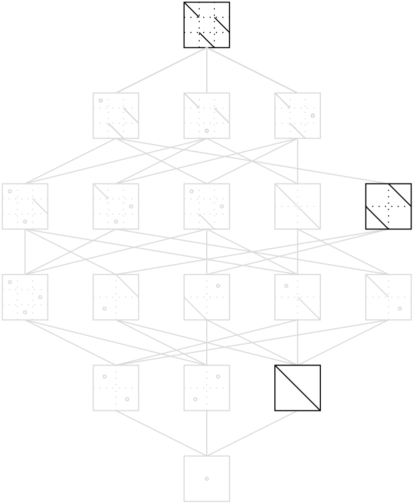

$$\frac{z^3}{(1-z)^3}$$

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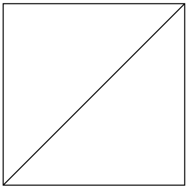
$$|\operatorname{Av}_n(123, 231)| = \frac{1}{2}(n^2 - n + 2)$$

Example: One Mutation



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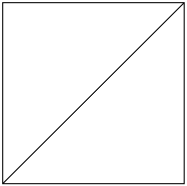

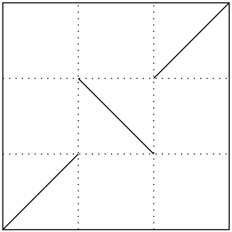

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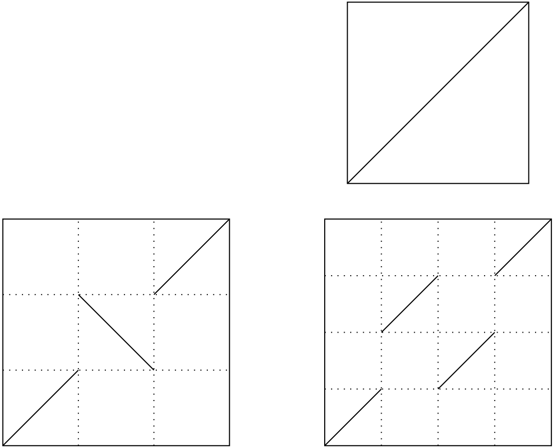

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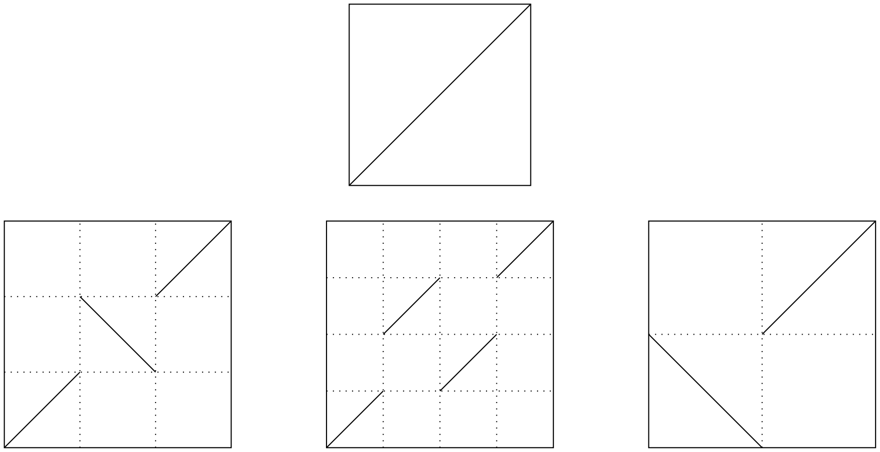
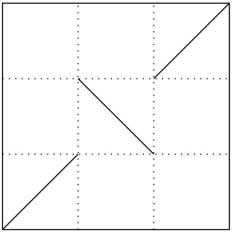
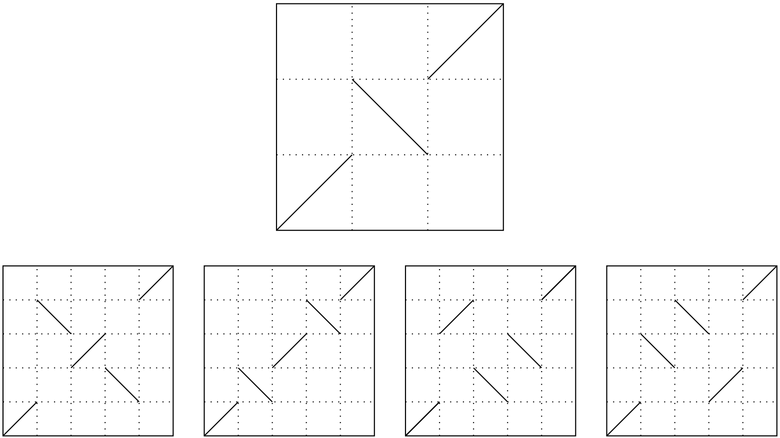


Two Block Reversals



Two Block Reversals



Computation

Example

```
>>> C = PolyClass.block_reversal(2)
>>>
>>> C.top_level
 {+1-3-4+2+5, +1+4-2-3+5, +1-2+3-4+5, +1-4+3-2+5}
>>>
>>> C.genfcn()
 -(x^7 - x^6 - 3*x^5 + 7*x^4 - 4*x^3 + 7*x^2 - 4*x + 1)/(x - 1)^5
>>>
>>> C.polynomial()
 8 + -19/6n^1 + 1/3n^2 + -1/3n^3 + 1/6n^4
>>>
>>> C.sequence(10)
 1, 1, 2, 6, 22, 63, 145, 288, 516, 857
```

Combinatorial Testing

Black Box Testing

Problem

How can you be sure that your new website/satellite/algorithm does what it's supposed to?

Black Box Testing

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(Bad) Solution

Try *every* input, and make sure nothing goes wrong.

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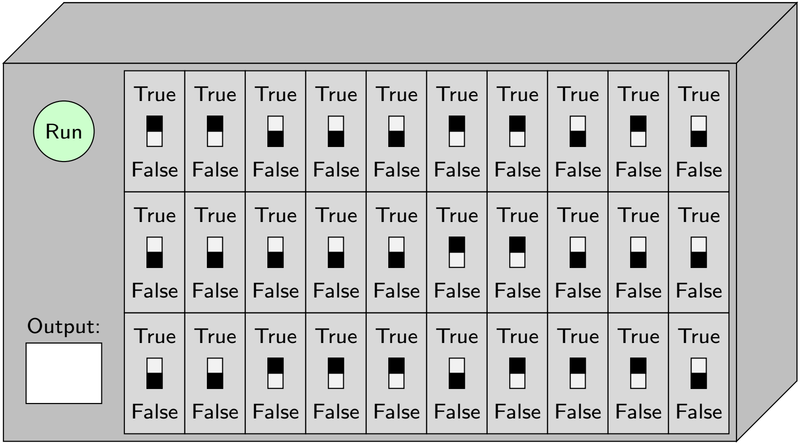
Try *every* input, and make sure nothing goes wrong.

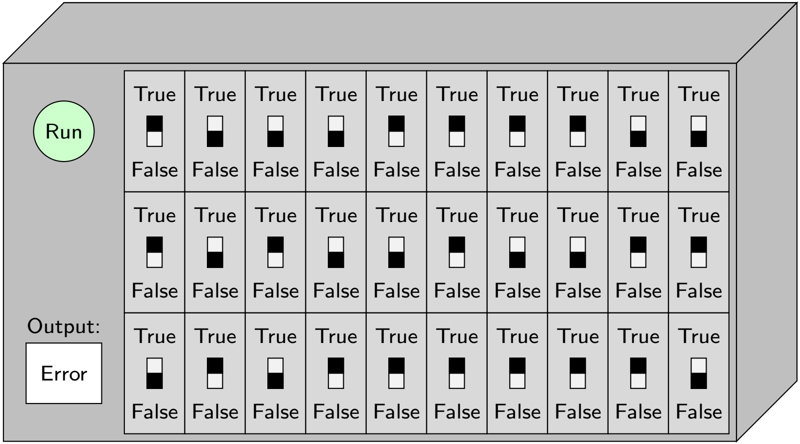
(Better) Solution

Try *some* inputs, and make sure nothing goes wrong.

Black Box Testing



Black Box Testing



$2^{30} > 10^9$ total input combinations

Parameter Interactions

Idea

Most errors are caused by interactions between relatively few parameters.

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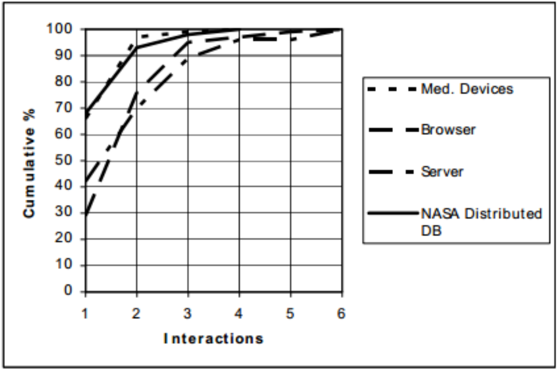

Source: NIST

Example: Printer

Example: Printer

Paper	Duplex	Color	Content	Orientation
A4	Yes	Yes	Text	Portrait
Letter	No	No	Pictures	Landscape
Legal			Both	

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$$3 \times 2 \times 2 \times 3 \times 2 = 56 \text{ combinations}$$

Example: Printer

Paper	Duplex	Color	Content	Orientation
A4	Yes	Yes	Text	Portrait
Letter	No	No	Pictures	Landscape
Legal			Both	

$$3 \times 2 \times 2 \times 3 \times 2 = 56 \text{ combinations}$$

Example

Suppose that printing two-sided color landscape pictures on legal paper breaks the printer...

Example: Printer Tests

	paper	duplex	color	content	orientation
1	a4	false	false	text	landscape
2	a4	true	true	pictures	portrait
3	a4	false	true	both	landscape
4	letter	true	false	text	portrait
5	letter	false	true	pictures	landscape
6	letter	true	false	both	portrait
7	legal	false	true	text	portrait
8	legal	true	false	pictures	landscape
9	legal	false	false	both	portrait

Example: Printer Tests

	P1	P2	P3	P4	P5
1	1	0	0	0	1
2	1	1	1	1	0
3	1	0	1	2	1
4	2	1	0	0	0
5	2	0	1	1	1
6	2	1	0	2	0
7	3	0	1	0	0
8	3	1	0	1	1
9	3	0	0	2	0

Covering Arrays

Covering Arrays

Problem

Represent a system with k parameters by a sequence $(p_1, p_2, \ldots p_k) \in \mathbb{N}^k$ where p_i denotes the number of variables for the ith parameter. (the printer is represented by $(3, 2, 2, 3, 2)$)

Want to construct a sequence of tests which covers every 2-way interaction between variables. That is, want an $l \times k$ array $\mathcal{A}$ such that, for every $1 \leq i < j \leq k$ and every pair $(v, w) \in \{1, \ldots p_i\} \times \{1, \ldots p_j\}$, there exists some $1 \leq r \leq l$ with

$$\mathcal{A}_{r,i} = v \text{ and } \mathcal{A}_{r,j} = w.$$

IPO Algorithm

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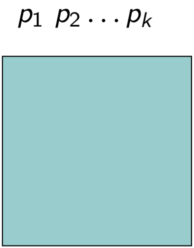

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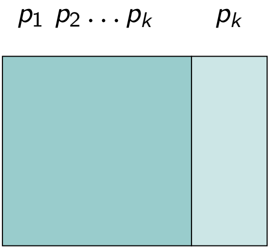
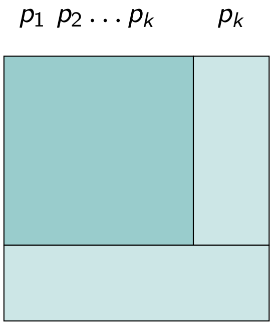


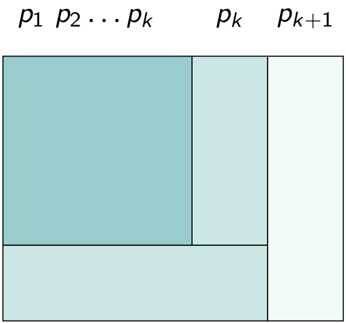
Horizontal Growth

IPO Algorithm



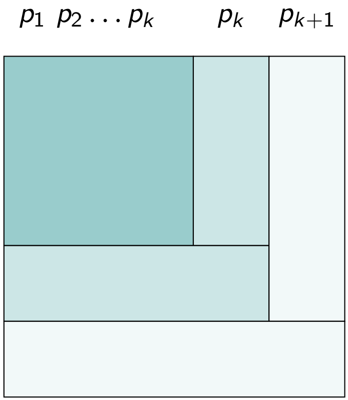
Vertical Growth

IPO Algorithm



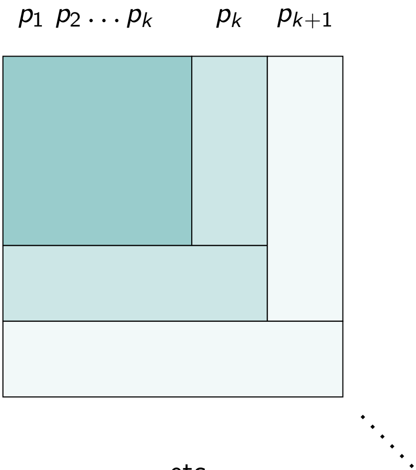
Horizontal Growth

IPO Algorithm



Vertical Growth

IPO Algorithm



etc.

IPO Algorithm

p_1	p_2
0	0
0	1
1	1
1	0

IPO Algorithm

p_1	p_2	p_3
0	0	0
0	1	1
1	1	0
1	0	1

IPO Algorithm

p_1	p_2	p_3	p_4
0	0	0	0
0	1	1	1
1	1	0	1
1	0	1	1

IPO Algorithm

p_1	p_2	p_3	p_4
0	0	0	0
0	1	1	1
1	1	0	1
1	0	1	1
1	1	1	0

IPO Algorithm

p_1	p_2	p_3	p_4	p_5
0	0	0	0	0
0	1	1	1	1
1	1	0	1	0
1	0	1	1	0
1	1	1	0	1

IPO Algorithm

p_1	p_2	p_3	p_4	p_5
0	0	0	0	0
0	1	1	1	1
1	1	0	1	0
1	0	1	1	0
1	1	1	0	1
1	0	0	0	1

IPOG Algorithm

Definition

A *strength t* covering array is one in which every t-way combination of variables appears in at least one row.

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Approximate Size

The number of tests is a minimal covering array of strength t for a system with n parameters, each of which having v variables, is

$$v^t \log(n).$$

Questions?

Thanks for listening!