Automatic Enumeration of Polynomial Permutation Classes and Applications to Genomics

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Permutation Patterns and Classes

Let $p = p_1 p_2 \ldots p_n$ be a permutation written in one-line notation, and let q be a permutation of length $k \leq n$.
We say that q is contained in p as a pattern (denoted $q \prec p$) if there is a sequence $1 \leq i_1, i_2, \ldots i_k \leq n$ so that the sequence of entries

$$p_{i_1} p_{i_2}, \ldots p_{i_n}$$

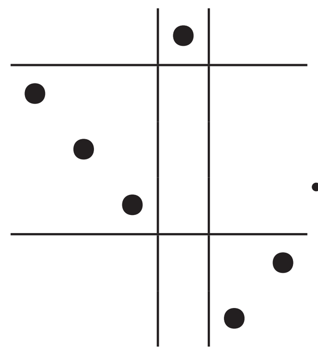
is in the same relative order as the entries of q.
A *permutation class* is a set $\mathcal{C}$ of permutations for which, if $p \in \mathcal{C}$ and $q \prec p$, $q \in \mathcal{C}$.

Peg Classes

A *peg matrix* is a square matrix M with entries $0, 1, -1, \bullet$, with exactly one nonzero entry per row and column. A peg matrix can be represented by a *peg permutation*, a permutation in which each entry carries either a 1, -1, or $\bullet$. We say that a permutation p is *griddable* by M if M can be overlaid onto the plot of p so that each 1 entry of M corresponds to an increasing run of p, each -1 to a decreasing run, $\bullet$ to at most a single entry, and 0 to an empty block.
Denote by $G(M)$ the class of permutations griddable by M. Say that a permutation $p \in G(M)$ *fills* the class if it has at least two entries in each 1 and -1 block, and exactly one in each $\bullet$ block.

Example

Let $M = \begin{pmatrix} 0 & \bullet & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then M can be represented as $\overset{-\bullet+}{231}$ and **543612** fills $G(M)$ since

543612 =  .

Theorem - Huczynska and Vatter

A permutation class is eventually enumerated by a polynomial if and only if it is contained in a peg class.

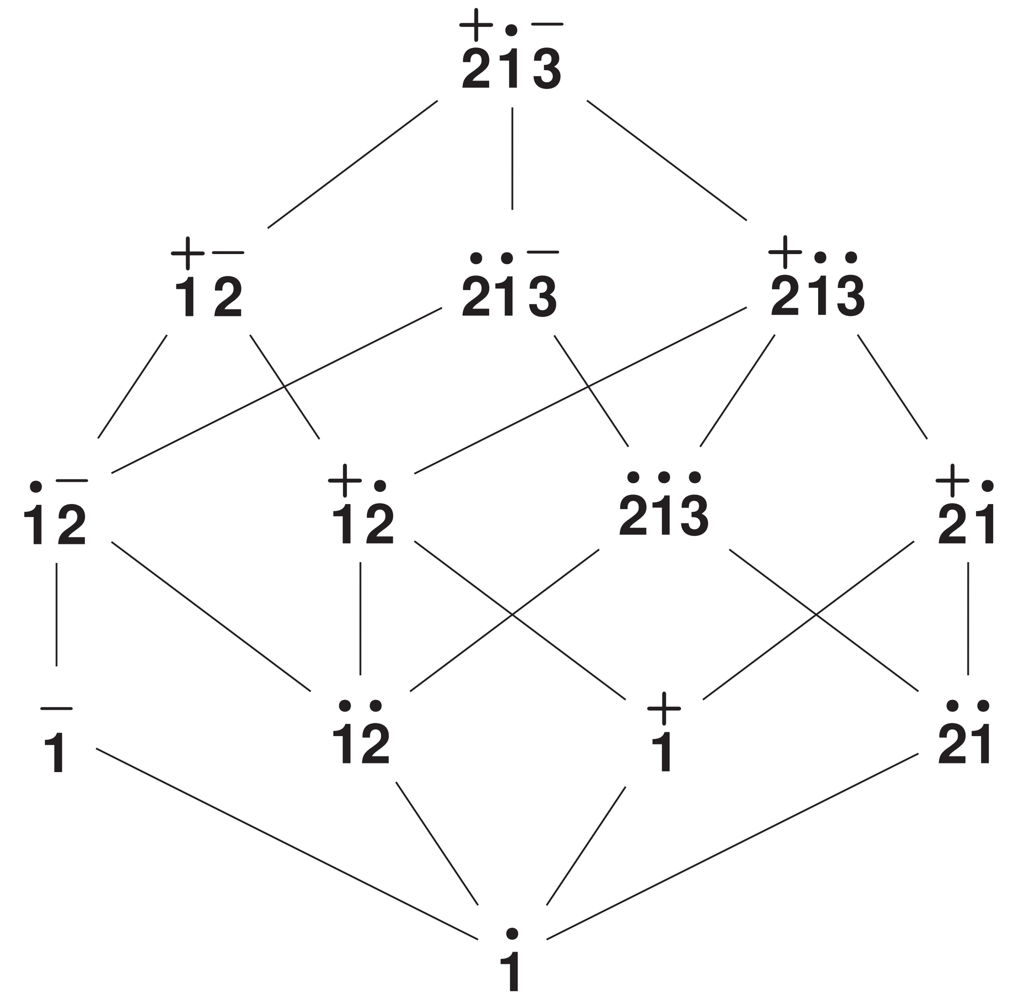
Theorem

If a class is contained in a peg class, it can be expressed as a finite union of peg classes.

Enumerating Polynomial Classes

The generating function for the number of permutations *filling* a peg classs is in general simple to compute. If $\mathcal{C}$ is a union of peg classes, we find a collection $\mathcal{P}$ of peg classes such that each permutation $p \in \mathcal{C}$ fills a unique peg class in $\mathcal{P}$, allowing us to easily enumerate the class $\mathcal{C}$.
We begin by generalizing pattern containment to apply to peg permutations, with the rule that $q \prec p$ as peg permutations if the underlying permutation of q is contained in p as a pattern, and each sign of q is either a dot or matches the corresponding sign of p.

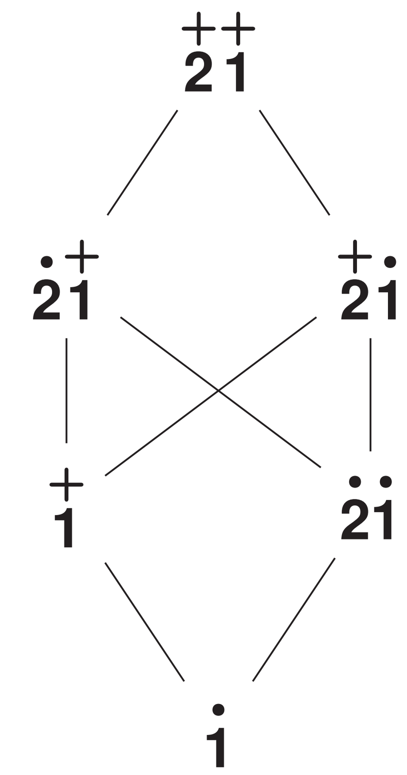
Example - The Downset of $\overset{+\bullet-}{213}$



Theorem

Say that a pattern q in a permutation p is *consecutive* if there are no entries of p in between the entries of q both vertically and horizontally.
Let $\mathcal{C}$ be a union of peg classes, and let S denote the union of downsets of each peg permutation in $\mathcal{C}$.
Remove all members of S which contain the consecutive patterns $\overset{++}{12}, \overset{+\bullet}{12}, \overset{\bullet+}{12}$, or $\overset{--}{21}, \overset{-\bullet}{21}, \overset{\bullet-}{21}$. Finally, if a peg permutation $q \in S$ can be reduced to another peg permutation in S by mapping the consecutive patterns $\overset{\bullet\bullet}{12} \mapsto \overset{+}{1}$ or $\overset{\bullet\bullet}{21} \mapsto \overset{-}{1}$, remove q from S.
Now, any permutation $\pi \in G(p)$ fills one and only one peg permutation in S.

Enumerating the Class $\overset{++}{21}$



$$\left(\frac{x^2}{1-x}\right)^2 + x\left(\frac{x^2}{1-x}\right) + \left(\frac{x^2}{1-x}\right)x + \left(\frac{x^2}{1-x}\right) + x^2 + x$$

$$= 1x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \ldots$$

$$G(\overset{++}{21}) = \{1, 12, 21, 123, 231, 312, \ldots\}.$$

Applications to Genomics

Genome evolution can be modeled using a variety of permutation block sorting operations (transposing or reversing contiguous blocks of entries). These methods can be easily represented with peg classes by applying the operations to the class $G(\overset{+}{1})$.
For example, the set of permutations which are at most one block transposition from the identity is given by the class $G(\overset{++++}{1324})$, while those that are at most two block reversals from the identity are

$$G(\overset{+-+-+}{12345}) \cup G(\overset{+-+-+}{14325}) \cup G(\overset{++--+}{14235}) \cup G(\overset{+--++}{13425}).$$

For length $n \geq 3$ their numbers, respectively, are

$$(n^3/6 - n/6 + 1) \text{ and } (n^4/6 - n^3/3 + n^2/3 - 19n/6 + 8).$$

References

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- Joint work with Vincent Vatter

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