Expected Patterns in Permutation Classes

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Rochester Institute of Technology

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Introduction

Example

The permutation 3 1 5 2 4 contains the pattern 3 1 2 twice (the first, second and forth entries and the third, forth and fifth).

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The set $\{2341, 1234, 4321\}$ contains the pattern 123 exactly 5 times.

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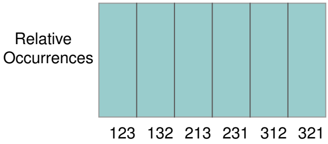

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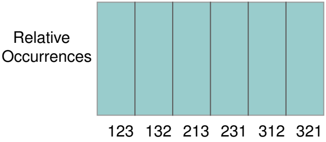
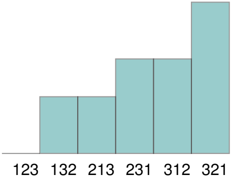


S_n



$\mathrm{Av}_n\,123$

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Similarly, for a pattern q and a set S of permutations, define

$$\mathrm{f}_q(S) = \sum_{p \in S} \mathrm{f}_q(p).$$

$$\mathrm{f}_{231}(S_n) = \frac{1}{n+1}\binom{2n}{n} = c_n$$

Previous Results

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Theorem (Bóna)

In $\mathrm{Av}_n 132$, the pattern 123 is the least common, 321 is the most common, and $\mathrm{f}_{213} = \mathrm{f}_{231} = \mathrm{f}_{312}$.

In addition, let q, t be any two non-empty patterns which end in their largest entry, and let i_u denote the increasing permutation of lenght u. Then

$$\mathrm{f}_{(q \ominus t) \oplus i_u} = \mathrm{f}_{(q \oplus i_u) \ominus t}$$

Data

Data

Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
4	10	0	11	11	11	13
5	68	0	81	81	81	109
6	392	0	500	500	500	748
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7	0	1578	1578	2794	2794	6271

Preliminaries

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$$\sum_{n\geq 0} \mathrm{f}_{12}(\mathrm{Av}_n\, 123)x^n = \frac{1-2x-\sqrt{1-4x}}{2(1-4x)}.$$

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$$(\mathrm{f}_{12}+\mathrm{f}_{21})(\mathrm{Av}_n\,123) = \binom{n}{2}c_n.$$

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Proof.

Rewrite as

$$(n-2)\,\mathrm{f}_{12} - \mathrm{f}_{132} - \mathrm{f}_{213} = \mathrm{f}_{231} + \mathrm{f}_{312} + \mathrm{f}_{132} + \mathrm{f}_{213}\,.$$

Both sides count the number of length three patterns with at least one non-inversion. □

Indecomposable Permutations

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Definition

We say that a permutation $p = p_1 p_2 \dots p_n$ is *decomposable* if there exists an integer k so that each of the entries $p_1, \dots p_k$ is greater than each of the entries $p_{k+1}, \dots p_n$. Otherwise, we say that p is *indecomposable*

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The permutation 356412 is decomposable, as the entries 3564 are larger than the entries 12

Definition

Denote by $\mathrm{Av}_n^* 123$ the set of indecomposable n-permutations which avoid 123.

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$|\operatorname{Av}_n^* 123| = c_{n-1}$.

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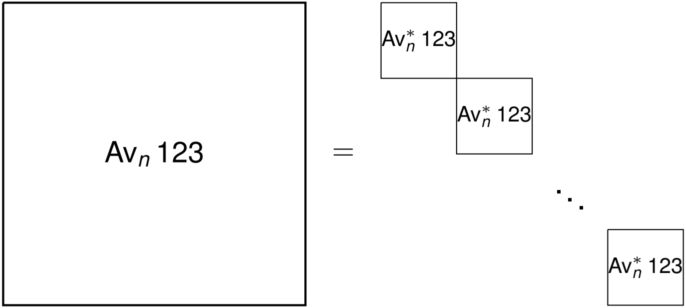

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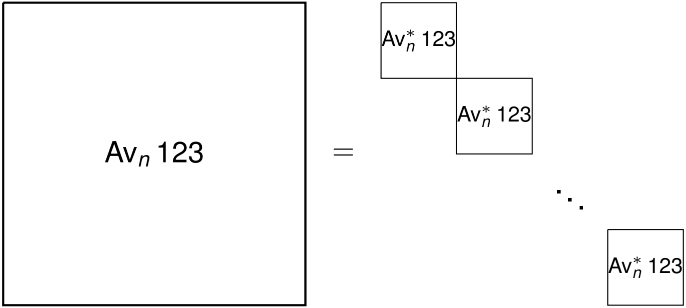

$$C(x) = 1 + C^*(x) + C^*(x)^2 + \ldots = \frac{1}{1 - C^*(x)}$$

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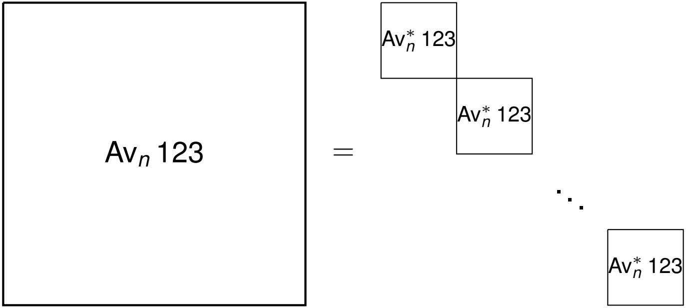

$$C(x) = 1 + C^*(x) + C^*(x)^2 + \ldots = \frac{1}{1 - C^*(x)}$$

$$C^*(x) = \frac{C(x) - 1}{C(x)} = xC(x).$$

□

Indecomposable Permutations

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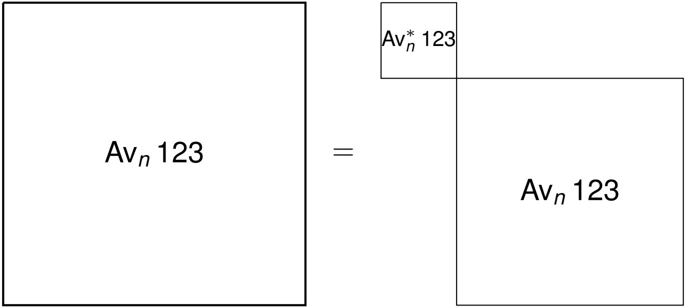
$|\operatorname{Av}^*_n 123| = c_{n-1}$.

Alternate Proof.



$$C(x) = C^*(x)C(x) + 1$$

$$C^*(x) = \frac{C(x) - 1}{C(x)} = xC(x).$$

□

Solving the System

Conjectures

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Conjectures

$$C(x)A(x) = xJ(x)C'(x)$$

$$A^*(x) + B^*(x) = \sum_{n \geq 0} \mathrm{f}_{213}\left(\mathrm{Av}_n^*\,132\right)x^n$$

$$B^*(x)C(x) = 2xB(x)$$

$$A(x) + B(x) = 2\sum_{n \geq 0}\Big(\mathrm{f}_{213}\left(\mathrm{Av}_n^*\,132\right) + \mathrm{f}_{231}\left(\mathrm{Av}_n^*\,132\right)\Big)x^n$$

$$A(x) + B(x) = xB^*(x)$$

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Solving the System

Corollary

$$C(x)A(x) = xJ(x)C'(x)$$

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$$A(x) + B(x) = xB^*(x)$$

$$J^*(x) = 2A^*(x)$$

Lemma

$$A^*(x) = \frac{x^3 C(x)}{(1-4x)^{3/2}}$$

One Last Definition

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Definition

A *Dyck path* of length $2n$ (or of semilength n) is a path in the plane from $(0, 0)$ to $(2n, 0)$ using steps $(1, 1)$ and $(1, -1)$ which never crosses below the x-axis.

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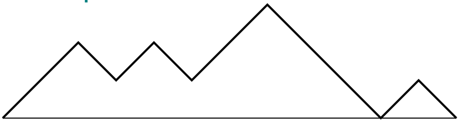 $= UUDUDUUDDDUD.$

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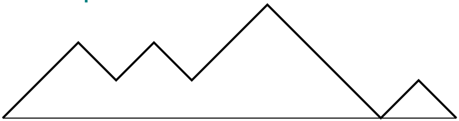 $= UUDUDUUDDDUD.$

Fact

There are exactly c_n Dyck paths of semilength n.

Sketch of proof

Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.

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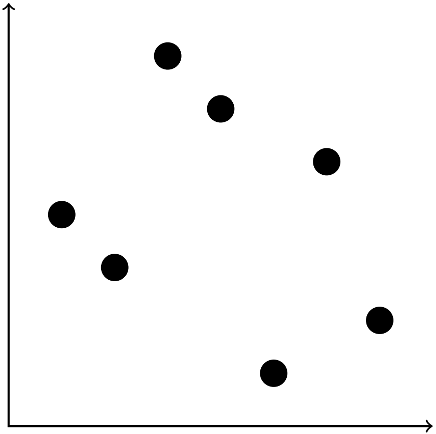

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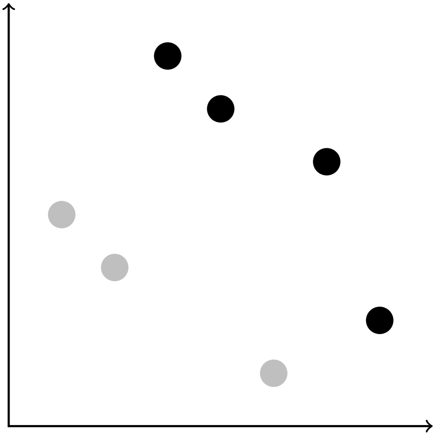

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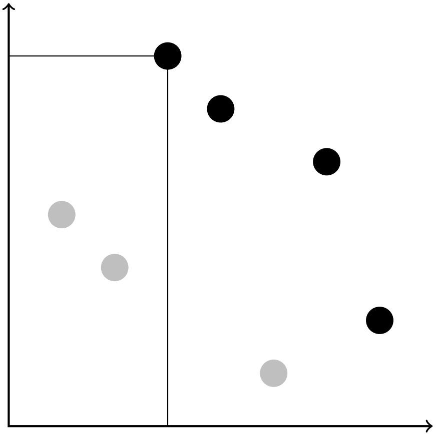

$$f_{213}(p) = \binom{2}{2}$$

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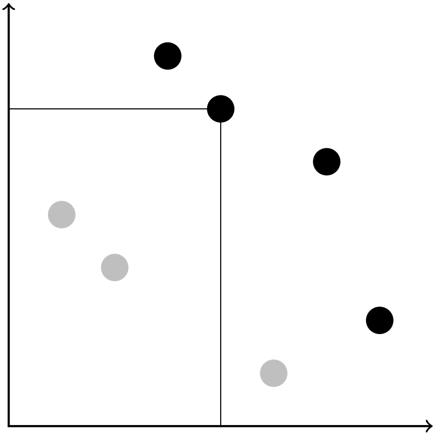

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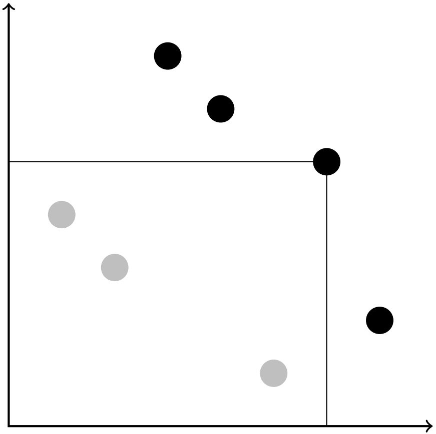

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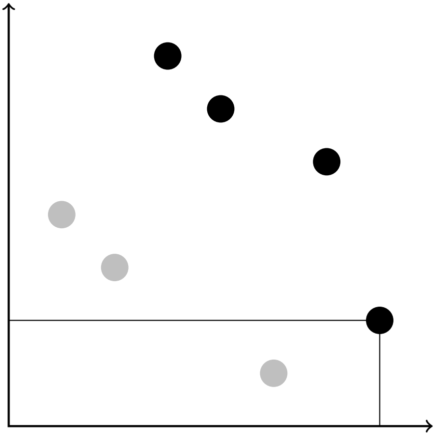

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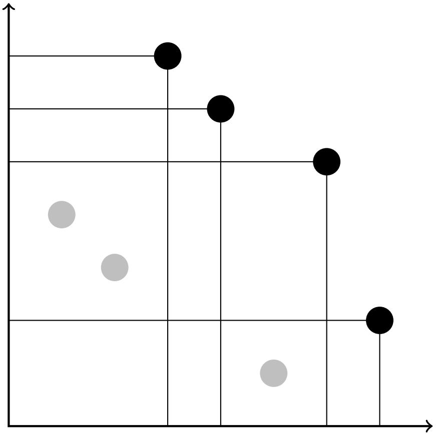

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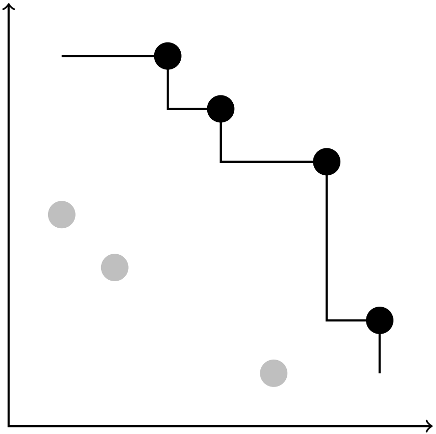

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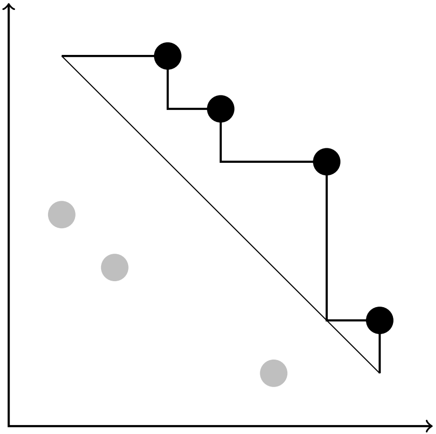

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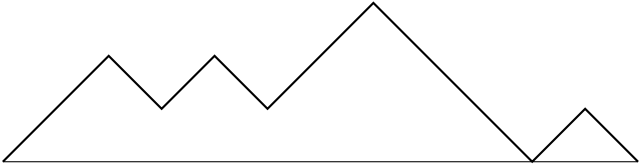

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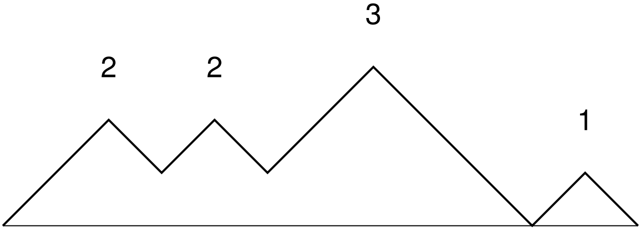

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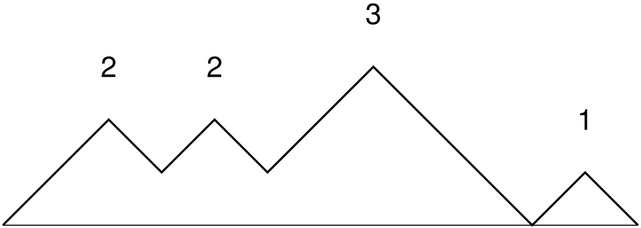
Let $h_{n,k}$ denote the total number of peaks at height k in all Dyck paths of semilength n. Let $H(x,u) = \sum_{n,k\geq 0} h_{n,k} x^n u^k$.



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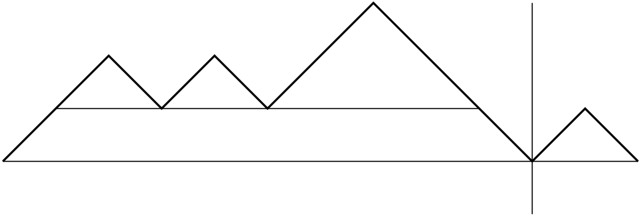

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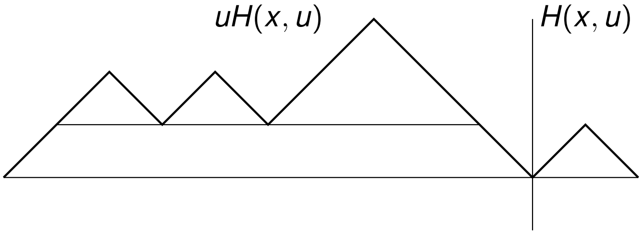

$$H(x,u) = ux(H(x,u)+1)C(x) + xC(x)H(x,u)$$

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□

Corollaries

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Corollary

$$\mathrm{f}_{231}(\mathrm{Av}_n\,123) = \mathrm{f}_{231}(\mathrm{Av}_n\,132)$$

Corollaries

$$A(x) = \frac{x-1}{2(1-4x)} - \frac{3x-1}{2(1-4x)^{3/2}}$$

$$B(x) = \frac{3x-1}{(1-4x)^2} - \frac{4x^2-5x+1}{(1-4x)^{5/2}}$$

$$D(x) = \frac{8x^3-20x^2+8x-1}{(1-4x)^2} - \frac{36x^3-34x^2+10x-1}{(1-4x)^{5/2}}$$

Corollaries

$$a_n = \frac{n+2}{4}\binom{2n}{n} - 3 \cdot 2^{2n-3}$$

$$b_n = (2n-1)\binom{2n-3}{n-2} - (2n+1)\binom{2n-1}{n-1} + (n+4) \cdot 2^{2n-3}$$

$$d_n = \frac{1}{6}\binom{2n+5}{n+1}\binom{n+4}{2} - \frac{5}{3}\binom{2n+3}{n}\binom{n+3}{2} + \frac{17}{3}\binom{2n+1}{n-1}\binom{n+2}{2} - 6\binom{2n-1}{n-2}\binom{n+1}{2} - (n+1) \cdot 4^{n-1}.$$

Corollaries

$$a_n \sim \sqrt{\frac{n}{\pi}} 4^n$$

$$b_n \sim \frac{n}{2} 4^n$$

$$d_n \sim \frac{8}{3} \sqrt{\frac{n^3}{\pi}} 4^n.$$

Larger patterns

Larger patterns

Lemma

$$\begin{aligned} 2A(x) + 2B(x) + D(x) &= \frac{x^3}{6}(C(x))''' \\ 4A(x) + 2B(x) &= x^3(J(x)/x^2)' \end{aligned}$$

Larger patterns

Lemma

$$
\begin{array}{lllll}
2A(x) & +2B(x) & +D(x) & = & \frac{x^3}{6}(C(x))''' \\
4A(x) & +2B(x) & & = & x^3(J(x)/x^2)'
\end{array}
$$

Theorem

For large enough n, the descending pattern of length k occurs more often than any other length k pattern in $\mathrm{Av}_n\,123$.

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No other ’surprising’ symmetries found for patterns of length 5 in Av 123, or for any patterns in Av q, for $|q| = 4$.

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